Scattering from infinity with singular asymptotics for wave equations satisfying the weak null condition.

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Let $\Box = -\partial_t^2 + \triangle_x$ be the wave operator in 3 space dimensions. Consider

$$\Box\,\varphi = (\partial_t \psi)^2, \qquad \Box\,\psi = 0,$$

Have a global solutions for $0 \leq t < \infty$ for given initial data when $t = 0$. We want to prescribe data at $t = \infty$ and solve the backwards problem. First we need to understand the asymptotics as $t \to \infty$.

For the linear homogeneous wave equation we have

$$\psi(t,x) \sim \mathcal{F}(r-t,\omega)/r, \qquad \text{where} \quad |\mathcal{F}(q,\omega)| + \langle q\rangle |\mathcal{F}_q(q,\omega)| \lesssim 1.$$

The same is true if only

$$|\Box\,\psi\,| + r^{-2}|\triangle_\omega \psi| \lesssim r^{-1}\langle t+r\rangle^{-1-\varepsilon}\langle t-r\rangle^{-1+\varepsilon}\langle (r-t)_+\rangle^{-\varepsilon}; \qquad \varepsilon > 0,$$

This is seen by expressing the wave operator in spherical coordinates:

$$\Box\,\psi = -r^{-1}(\partial_t + \partial_r)(\partial_t - \partial_r)(r\psi) + r^{-2}\triangle_\omega \psi,$$

and integrating, in the $t+r$ direction and in the $t-r$ direction.

However, general quadratic terms do not decay enough for this to hold:

$$\psi_t(t,x)^2 \sim \mathcal{F}_q'(r-t,\omega)^2/r^2.$$

The asymptotics for the wave equation with such sources along light cones

$$-\Box\,\phi = n(r-t,\omega)/r^2, \qquad |n(q,\omega)| \lesssim \langle q\rangle^{-1-\epsilon}, \quad \epsilon > 0$$

The solution to the forward problem has a log correction in the asymptotics

$$\phi(t,r\omega) \sim \ln\Big|\frac{r}{\langle t-r\rangle}\Big|\,\frac{\mathcal{F}_{01}(r-t,\omega)}{r} + \frac{\mathcal{F}_0(r-t,\omega)}{r}, \qquad \text{as} \quad t\to\infty, \quad r\sim t,$$

In fact, using the expression for the wave operator in spherical coordinates

$$\Box\phi(t,r\omega) \sim 2\mathcal{F}_{01,q}'(r-t,\omega)/r^2 + O\big(1/r^3\big) \sim n(r-t,\omega)/r^2,$$

if

$$2\mathcal{F}_{01,q}'(q,\omega) = n(q,\omega).$$

This only determines $\mathcal{F}_{01}(q,\omega)$ up to a function of ω

$$\lim_{q\to-\infty} \mathcal{F}_{01}(q,\omega) = N_{01}(\omega),$$

that has be determined from interior homogeneous asymptotics.
$\mathcal{F}_0(q,\omega)$ is free to chose and determined from initial data apart from that it has to match interior asymptotics

$$\lim_{q\to-\infty} \mathcal{F}_0(q,\omega) = N_0(\omega),$$

Interior asymptotics for the wave equation with such sources on light cones

$$-\Box\,\phi = n(r-t,\omega)/r^2, \qquad |n(q,\omega)| \lesssim \langle q\rangle^{-1-\epsilon}, \quad \epsilon>0.$$

The forward problem for this equation has homogeneous asymptotics

$$\phi(t,x)\sim\phi_\infty(t,x)=\Psi(x/t)/t, \qquad t>|x|, \quad t\to\infty.$$

In fact, $\phi_a(t,x)=a\,\phi(at,ax)$ satisfies

$$-\Box\,\phi_a = n_a(r-t,\omega)/r^2, \qquad n_a(q,\omega)=a\,n(aq,\omega).$$

As $a\to\infty$, in the sense of distribution theory

$$n_a(q,\omega)=a\,n(aq,\omega)\to\delta(q)\Phi(\omega), \qquad \text{where} \quad N(\omega)=\int_{-\infty}^{+\infty} n(q,\omega)\,dq,$$

and $\delta(q)$ is the delta function. Hence $\phi_a\to\phi_\infty$ where

$$-\Box\,\phi_\infty = N(\omega)\delta(r-t)/r^2.$$

Since this is homogeneous of degree -3, ϕ_∞ is homogeneous of degree -1. We claim that ϕ_∞ has the asymptotics as we approach the light cone:

$$\phi_\infty(t,r\omega)\sim \ln\Big|\tfrac{r}{t-r}\Big|N_{01}(\omega)/r + N_0(\omega)/r, \qquad r\to t.$$

In fact convolving with the fundamental solution of $\Box$ gives a formula

$$\phi_\infty(t,r\omega)=\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{N(\sigma)\,dS(\sigma)}{t-\langle\sigma,r\omega\rangle}.$$

Higher order asymptotics, in the wave zone $r \sim t$:

$$\phi(t, r\omega) \sim \Psi_{rad}(r-t, \omega, 1/r)$$
$$= \ln\left|\frac{2\,r}{\langle t-r\rangle}\right| \frac{\mathcal{F}_{01}(r-t,\omega)}{r} + \frac{\mathcal{F}_0(r-t,\omega)}{r} + \ln\left|\frac{2\,r}{\langle t-r\rangle}\right| \frac{\mathcal{F}_{11}(r-t,\omega)}{r^2} + \frac{\mathcal{F}_1(r-t,\omega)}{r^2},$$

and in the interior $r < t$:

$$\phi_\infty(t, r\omega) \sim \Psi_{hom}(r-t, \omega, 1/r)$$
$$= N_{01}(\omega)\frac{1}{r}\ln\left|\frac{2\,r}{t-r}\right| + N_0(\omega)\frac{1}{r} + N_{11}(\omega)\frac{r-t}{r^2}\ln\left|\frac{2\,r}{t-r}\right| + N_1(\omega)\frac{r-t}{r^2}.$$

Matching conditions

$$\lim_{q\to-\infty} \mathcal{F}_j(q,\omega) = N_j(\omega), \qquad \lim_{q\to-\infty} \mathcal{F}_{j1}(q,\omega) = N_{j1}(\omega), \quad j = 0, 1.$$

Assume that

$$|\,(\langle q\rangle\partial_q)^k \partial_\omega^\alpha n(q,\omega)| \le C\langle q\rangle^{-2-2\gamma}, \qquad k + |\alpha| \le N, \quad 0 < \gamma < 1.$$

Construct approximate solution. Let $\chi_a(t,x) = \chi\big((r-t)/r^a\big)$, where $\chi(s)=1$, when $s\geq -1/2$ and $\chi(s)=0$ when $s\leq -1$, $0<a<1$. Set

$$\Psi_{app} = \chi_a \Psi_{rad} + (1-\chi_a)\Psi_{hom},$$

where

$$\Box\,\Psi_{rad} \sim -n(r-t,\omega)/r^2, \qquad \Box\,\Psi_{hom} = 0$$

With $\Psi_{diff} = \Psi_{rad} - \Psi_{hom}$ we have

$$\Box\Psi_{app} = \chi_a \Box\Psi_{rad} + (1-\chi_a)\Box\Psi_{hom} + \Box\chi_a\,\Psi_{diff} + 2Q(\partial\chi_a, \partial\Psi_{diff})$$

$$\Box\Psi_{app} + n(r-t,\omega)/r^2 \sim (1-\chi_a)n(r-t,\omega)/r^2 + \Box\chi_a\,\Psi_{diff} + 2Q(\partial\chi_a,\partial\Psi_{diff})$$

With $\mathcal{H}_{j1} = \mathcal{F}_{j1} - N_{j1}$ and $\mathcal{H}_j = \mathcal{F}_j - N_j$, for $j=0,1$ we have

$$\Psi_{diff}(t,r\omega)$$
$$= \ln\Big|\frac{2\,r}{\langle t-r\rangle}\Big|\frac{\mathcal{H}_{01}(r-t,\omega)}{r} + \frac{\mathcal{H}_0(r-t,\omega)}{r} + \ln\Big|\frac{2\,r}{\langle t-r\rangle}\Big|\frac{\mathcal{H}_{11}(r-t,\omega)}{r^2} + \frac{\mathcal{H}_1(r-t,\omega)}{r^2}$$

Because of the matching $\mathcal{H}$ decays more in q

$$|\mathcal{H}_0| + \langle q\rangle^{-1}|\mathcal{H}_1| + |\mathcal{H}_{01}| + \langle q\rangle^{-1}|\mathcal{H}_{11}|$$
$$+ \langle q\rangle\big(|\mathcal{H}'_{0,q}| + \langle q\rangle^{-1}|\mathcal{H}'_{1,q}| + |\mathcal{H}'_{01,q}| + \langle q\rangle^{-1}|\mathcal{H}'_{11,q}|\big) \lesssim \langle q\rangle^{-\gamma}.$$

and that improves the decay in the transition region $r-t\sim -r^a$.

With the appropriate choice of $0 < a < 1$ we get

$$\Box \Psi_{app} + n(r - t, \omega)/r^2 = F$$

where F decays fast so we can find a solution of

$$\Box \Psi_{err} = -F, \quad \text{with} \quad |\Psi_{err}| \lesssim \langle t \rangle^{-1-b}, \quad b > 0.$$

Hence $\Psi = \Psi_{app} + \Psi_{err}$ is a solution to

$$\Box \, \Psi = -n(r - t, \omega)/r^2.$$