Integration

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For all the problems, check [2].

1. Compute

\[ I_1 = \int \frac{\sin x}{\sin x + \cos x} \, dx, \quad I_2 = \int \frac{\cos x}{\sin x + \cos x} \, dx \]

2. For \( a > 0 \) compute the integral

\[ \int \frac{1}{x\sqrt{x^2 + x^a + 1}} \, dx, \quad x > 0 \]

3. 

\[ \int (1 + 2x^2)e^x \, dx \]

4. 

\[ \int \frac{x + \sin x - \cos x - 1}{x + e^x + \sin x} \, dx \]

5. 

\[ \int \frac{x^2 + 1}{x^4 - x^2 + 1} \, dx \]

Definite Integrals

1. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function. Prove that

\[ \int_0^\pi x f(\sin x) \, dx = \pi \int_0^\frac{\pi}{2} f(\sin x) \, dx. \]

2. 

\[ \int_{-1}^1 \frac{\sqrt{x}}{\sqrt{1 - x} + \sqrt{1 + x}} \, dx \]

3. 

\[ \int_0^\pi \frac{x \sin x}{1 + \sin^2 x} \, dx \]

4. Let \( a \) and \( b \) positive real numbers. Compute

\[ \int_a^b \frac{e^{x/a} - e^{b/x}}{x} \, dx \]
5. 
\[ \int_0^1 \frac{\ln(1 + x)}{1 + x^2} \, dx \]

6. 
\[ \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}, a > 0 \]

7. 
\[ \int_0^\infty \frac{\ln x}{x^2 + a^2} \, dx \]

Enumerative Combinatorics[1]

1. 2n players are participating in a tennis tournament. Find the number \( P_n \) of pairings for the first round.

2. We draw all diagonals of a convex \( n \)-gon. What is the number of intersection points of the diagonals? Suppose no three diagonals pass through a point. Into how many parts is the \( n \)-gon divided?

3. How many \( n \)-words from the alphabet \{0, 1, 2\} are such that neighbors differ at most by 1?

4. Each of the faces of a cube is colored by a different color. How many of the colorings are distinct?

5. In how many ways can you take an odd number of objects from \( n \) objects?

6. 2n objects of each of three kinds are given to two persons, so that each person gets 3n objects. Prove that this can be done in \( 3n^2 + 3n + 1 \) ways.

References
