Mathematical Induction

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1 Introduction

Example 1: Suppose $F_n$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$.

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

Example 2: Find the exact value of the following expression

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

Example 3: (Fermat’s little theorem) Let $p$ be a prime number, and $n$ a positive integer. Then $n^p - n$ is divisible by $p$.

Example 4: (Arithmetic-mean-inequality) Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Show that

$$\frac{a_1 + \cdots + a_n}{n} \geq \left( a_1 \cdots a_n \right)^{\frac{1}{n}}$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2 Practise Problems

2.1 Part I

1. Prove that $(n+1)(n+2)\cdots2n = 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$ for all $n \in \mathbb{N}$.

2. Prove that $|\sin nx| \leq n|\sin x|$ for any real number $x$ and positive integer $n$.

3. Prove that $3^n > n^3$ for all positive integers $n$.

4. Let $n \geq 6$ be an integer, prove that

$$\left( \frac{n}{3} \right)^n < n! < \left( \frac{n}{2} \right)^n$$

5. Find the exact value of

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\cdots}}}$$
6. Suppose $F_n$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Show that

$$F_{2n} = F_n^2 + 2F_{n-1}F_n, \quad F_{2n-1} = F_{n-1}^2 + F_n^2.$$ 

7. $3^{n+1} | 2^{3^n} + 1$ for all integers $n \geq 0$.

8. Suppose $\alpha + 1/\alpha \in \mathbb{Z}$ for some real number. Prove that

$$\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}$$

for any $n \in \mathbb{N}$.

### 2.2 Part II

1. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with $1 \times 1$ corner square removed can be tiled by pieces of the following form

![Checkerboard Piece](image)

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(\frac{x_1 + x_2}{2}) = \frac{f(x_1) + f(x_2)}{2}$ for any $x_1, x_2$. Show that

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

3. Given a sequence of integers $x_1, x_2, \ldots, x_n$ whose sum is 1, prove that exactly one of the cyclic shifts

$$x_1, x_2, \cdots, x_n; \quad x_2, \cdots, x_n, x_1; \quad \cdots; \quad x_n, x_1, \cdots, x_{n-1}$$

has all of its partial sum positive. (By a partial sum we mean the sum of the first $k$ terms, $k \leq n$.)

4. Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some positive integer $n$ and some choice of the signs.

5. Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for every relatively prime pair of positive integers $m$ and $n$. Prove that $f(n) = n$ for every positive integer $n$.

6. Let $a_1, a_2, \cdots, a_n$ be real numbers greater than 1. Prove the inequality

$$\sum_{i=1}^{n} \frac{1}{1 + a_i} \geq \frac{n}{1 + \sqrt[n]{a_1a_2\cdots a_n}}.$$