1 Problems (all from [1])

**Fundamental theorem of algebra:** Every nonconstant polynomial with complex coefficients has at least one complex zero.

1. Find a polynomial with integer coefficients that has the zero $\sqrt{2} + \sqrt[3]{3}$.

2. Verify the identity
   \[
   \sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}} = 4.
   \]

3. Given the polynomial $P(x, y, z)$ prove that the polynomial
   \[
   Q(x, y, z) = P(x, y, z) + P(y, z, x) + P(z, x, y) - P(x, z, y) - P(y, x, z) - P(z, y, x)
   \]
   is divisible by $(x - y)(y - z)(z - x)$.

4. Find all polynomials satisfying the functional equation
   \[
   (x + 1)P(x) = (x - 10)P(x + 1).
   \]

5. Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$, counted without multiplicities.

6. Let $P(x)$ be a polynomial of degree $n$. Knowing that
   \[
   P(k) = \frac{k}{k + 1}, \quad k = 0, 1, \ldots, n
   \]
   find $P(m)$ for $m > n$.

7. Let $Q_0(x) = 1$, $Q_1(x) = x$, and
   \[
   Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}
   \]
   for all $n \geq 2$. Show that, whenever $n$ is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients. (Putnam A.2 in 2017.)

**Irreducible Polynomials** A polynomial is irreducible if it cannot be written as a product of two polynomials in a nontrivial manner. For matters of elegance we focus on polynomials with integer coefficients.

**Theorem (Eisenstein):** Given a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with integer coefficients, suppose that there exists a prime number $p$ such that $a_n$ is not divisible by $p$, $a_k$ is divisible by $p$ for $k = 0, 1, \cdots, n - 1$, and $a_0$ is not divisible by $p^2$. Then $P(x)$ is irreducible over $\mathbb{Z}[x]$. 
1. Prove that the polynomial $P(x) = x^{101} + 101x^{100} + 102$ is irreducible over $\mathbb{Z}[x]$.

2. Prove that for every prime number $p$, the polynomial $P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible over $\mathbb{Z}[x]$.

3. Prove that for every positive integer $n$, the polynomial $P(x) = x^{2^n} + 1$ is irreducible over $\mathbb{Z}[x]$.

4. Prove that for any distinct integers $a_1, a_2, \ldots, a_n$ the polynomial
   \[ P(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1 \]
   is irreducible over $\mathbb{Z}[x]$.

5. Let $P(x)$ be an $n$th-degree polynomial with integer coefficients with the property that $|P(x)|$ is a prime number for $2n + 1$ distinct integer values of the variable $x$. Prove that $P(x)$ is irreducible over $\mathbb{Z}[x]$.

**The Derivative of a Polynomial**

If a zero of $P(x)$ has multiplicity greater than 1, then it is also a zero of $P'(x)$, and the converse is also true.

1. Find all polynomials $P(x)$ with integer coefficients satisfying $P(P'(x)) = P'(P(x))$ for all $x \in \mathbb{R}$.

2. Determine all polynomials $P(x)$ with real coefficients satisfying $(P(x))^n = P(x^n)$ for all $x \in \mathbb{R}$, where $n > 1$ is a fixed integer.

3. Let $P(z)$ and $Q(z)$ be polynomials with complex coefficients of degree greater than or equal to 1 with the property that $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Prove that the polynomials are equal.

**References**