Series

Liming Sun

A series is a sum
\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \]

Geometric series
\[ 1 + x + x^2 + \cdots + x^n + \cdots \]
which converge if \(|x| < 1\) and diverge otherwise.

\(p\)-series
\[ 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \]
which converge if \(p > 1\) and diverge otherwise.

1. Let \(a_1, a_2, \ldots, a_n, \ldots\) be nonnegative numbers. Prove that \(\sum_{n=1}^{\infty} a_n\) converges implies \(\sum_{n=1}^{\infty} \sqrt{a_n} a_{n+1}\) converges.

2. Show that the series
\[ \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \cdots + \frac{2^n}{1+x^{2^n}} + \cdots \]
converges when \(|x| > 1\), and in this case find its sum.

3. For what positive \(x\) does the series converge?
\[(x-1) + (\sqrt{x}-1) + (\sqrt[3]{x}-1) + \cdots + (\sqrt[n]{x}-1)\]

4. Prove the identity \(\sum_{k=1}^{n} (k^2 + 1)k! = n(n+1)!\).

Telescopic Series

1. Find the sum
\[ \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}} \]

2. Find sum \(\sum_{n=1}^{49} \frac{1}{\sqrt{n}+\sqrt{n+1}}\).

3. Define the sequence \((a_n)_n\) by \(a_0 = 3\) and \(a_{n+1} = a_0 a_1 \cdots a_n + 2, n \geq 0\). Prove that
\[ a_{n+1} = 2(a_0 - 1)(a_1 - 1) \cdots (a_n - 1) + 1, \quad \text{for all } n \geq 0 \]

All problems are from [1].

References