Section 7.8: Repeated eigenvalues

In our study of systems of two linear first order ODEs

\[ \dot{\mathbf{x}} = A \mathbf{x}, \]

we have one case left when \( A \) is a real \( 2 \times 2 \) matrix:

It can happen that \( A \) has a repeated (real) eigenvalue which does not have enough linearly independent eigenvectors (i.e. the geometric multiplicity of the eigenvalue is strictly less than its algebraic multiplicity).

In this case we have to come up with a method to obtain another linearly independent solution to \( \dot{\mathbf{x}} = A \mathbf{x} \).

Example:

Find a fundamental set of solutions to

\[ \dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{x}, \]

\[ \mathbf{e} = A \]
Solution:

Characteristic equation:

\[ 0 = \det (A - r \cdot I_{mxr}) \]

\[ = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (1-r)(3-r) + 1 \]

\[ = r^2 - 4r + 4 = (r-2)^2 \]

\( \Rightarrow r_1 = r_2 = 2 \) is a repeated eigenvalue of \( A \).

Eigenvector(s) for \( r_1 = r_2 = 2 \):

\[ (A - 2 \cdot I_{mxr}) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \]

\[ \Rightarrow \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \]

The first row is \((-1)\) times the second row, thus we only get the constraint

\[ \xi_1 + \xi_2 = 0 \]

Say \( \xi_1 = \alpha \), then \( \xi_2 = -\alpha \)

\[ \Rightarrow \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

We only get one linearly independent eigenvector \( \xi \) for the eigenvalue 2.

Thus, \( \vec{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \) is a solution, but we need to find another linearly independent solution \( \vec{x}(t) \) to obtain a fundamental set of solutions to the system.
Compare with the situation for single variable 2nd order linear ODEs
\[ a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \]
whose characteristic equation has a repeated eigenvalue \( r \):

Then \( y_1(t) = e^{rt} \) and \( y_2(t) = te^{rt} \)
are a fundamental set of solutions.
Maybe this also works here?

Let's try the ansatz for another solution
\[ x^{(2)}(t) = \bar{z}^2 t e^{rt} \]
for a constant vector \( \bar{z}^2 e^{rt} \) to be determined:

Then we must have

\[
\frac{d}{dt}(x^{(2)}(t)) = \bar{z}^2 e^{rt} + 2\bar{z} \bar{z} t e^{rt} \\
A \cdot x^{(2)}(t) = A \bar{z}^2 t e^{rt}
\]

\[
\Rightarrow \quad \bar{z}^2 e^{rt} + 2\bar{z} \bar{z} t e^{rt} = A \bar{z}^2 t e^{rt}
\]
Divide by \( e^{rt} \)

\[
\Rightarrow \quad \bar{z}^2 = (A \bar{z}^2 - 2\bar{z}) \cdot t \quad \text{for all } t \text{ in } \mathbb{R}
\]
not possible.

When we computed \( \frac{d}{dt}(\bar{z}^2 t e^{rt}) \) we got terms with the factor \( t e^{rt} \) and terms just with the factor \( e^{rt} \). This is too much for a single undetermined vector \( \bar{z}^2 \) to take into account.
Let's therefore try the ansatz

\[ x^{(2)}(t) = \frac{\mathbf{\hat{c}}}{3} e^{2t} + \frac{\mathbf{\hat{c}}}{2} e^{2t}, \]

where \( \frac{\mathbf{\hat{c}}}{3}, \frac{\mathbf{\hat{c}}}{2} \epsilon \mathbb{R}^2 \) are constant vectors to be determined.

Then we have

\[ \frac{d}{dt} \left( x^{(2)}(t) \right) = \frac{\mathbf{\hat{c}}}{3} e^{2t} + 2 \frac{\mathbf{\hat{c}}}{3} t e^{2t} + 2 \frac{\mathbf{\hat{c}}}{2} e^{2t} \]

\[ A x^{(2)}(t) = A \frac{\mathbf{\hat{c}}}{3} e^{2t} + A \frac{\mathbf{\hat{c}}}{2} e^{2t} \]

\[ \frac{d}{dt}(x^{(2)}(t)) = A x^{(2)}(t) \]

\[ \Rightarrow \frac{\mathbf{\hat{c}}}{3} e^{2t} + 2 \frac{\mathbf{\hat{c}}}{3} t e^{2t} + 2 \frac{\mathbf{\hat{c}}}{2} e^{2t} = A \frac{\mathbf{\hat{c}}}{3} e^{2t} + A \frac{\mathbf{\hat{c}}}{2} e^{2t} \]

\[ \Rightarrow 0 = (A \frac{\mathbf{\hat{c}}}{3} - 2 \frac{\mathbf{\hat{c}}}{3}) t + A \frac{\mathbf{\hat{c}}}{2} - 2 \frac{\mathbf{\hat{c}}}{2} - \frac{\mathbf{\hat{c}}}{3} \]

for all \( t \).

We can achieve this for all times \( t \) if we have that

\[ (A - 2I_{2\times2}) \frac{\mathbf{\hat{c}}}{3} = 0 \]

\[ (A - 2I_{2\times2}) \frac{\mathbf{\hat{c}}}{2} = \frac{\mathbf{\hat{c}}}{2} \]

Here \( \frac{\mathbf{\hat{c}}}{2} \) is called a generalized eigenvector of \( A \):

\[ (A - 2I_{2\times2}) \frac{\mathbf{\hat{c}}}{2} = 0 \]

Thus, we may take

\[ \frac{\mathbf{\hat{c}}}{3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

and we compute \( \frac{\mathbf{\hat{c}}}{2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) to be
\[
\begin{bmatrix}
-1 & -1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1 \\
\end{bmatrix}
\]

\[\Rightarrow \alpha + \beta = -1 \]

Say \(\alpha = \lambda \in \mathbb{R}\), then \(\beta = -1 - \alpha\)

\[\Rightarrow \begin{bmatrix}
\alpha \\
\beta \\
\end{bmatrix}
= \begin{bmatrix}
\lambda \\
-1 - \lambda \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
-1 \\
\end{bmatrix} + \lambda \cdot \begin{bmatrix}
1 \\
-1 \\
\end{bmatrix}
\]

Thus, for the second solution we find

\[
\begin{aligned}
\ddot{x}^{(2)}(t) &= \frac{\alpha}{3} t e^{2t} + \frac{\beta}{2} e^{2t} \\
&= \begin{bmatrix}
1 \\
-1 \\
\end{bmatrix} t e^{2t} + \left(\begin{bmatrix}
0 \\
-1 \\
\end{bmatrix} + \lambda \begin{bmatrix}
1 \\
-1 \\
\end{bmatrix}\right) e^{2t}
\end{aligned}
\]

We can take

\[\ddot{x}^{(2)}(t) = \begin{bmatrix}
1 \\
-1 \\
\end{bmatrix} t e^{2t} + \begin{bmatrix}
0 \\
-1 \\
\end{bmatrix} e^{2t}\]

\[
\begin{aligned}
\text{Check that} &\quad W(\dot{x}^{(1)}(t), \dot{x}^{(2)}(t)) = - e^{4t} \neq 0 \text{ for all } t \in \mathbb{R} \\
\Rightarrow & \quad \dot{x}^{(1)}(t) \text{ and } \dot{x}^{(2)}(t) \text{ are linearly independent solutions and the general solution is}
\end{aligned}
\]

\[
\dot{x}(t) = c_1 \begin{bmatrix}
1 \\
-1 \\
\end{bmatrix} e^{2t} + c_2 \cdot \left(\begin{bmatrix}
1 \\
-1 \\
\end{bmatrix} t e^{2t} + \begin{bmatrix}
0 \\
-1 \\
\end{bmatrix} e^{2t}\right)
\]

\[
\dot{x}(t) = \dot{x}^{(1)}(t) = \dot{x}^{(2)}(t)
\]

For drawing the phase portrait, observe that

\[\lim_{t \to \infty} \frac{\dot{x}_2(t)}{\dot{x}_1(t)} = -1\]

which is the "slope of the eigenvector \(\frac{\dot{x}}{x}\) direction".
Phase portrait:

- straight line motion only along the single eigenvector
- origin is an equilibrium solution and is called an improper node
- here the origin is unstable (the repeated eigenvalue is positive; if the repeated eigenvalue is negative, then the origin is stable)
Summarizing:

Consider \( \dot{\mathbf{x}} = A \mathbf{x} \), where \( A \) is a real \( 2 \times 2 \) matrix with a repeated eigenvalue \( r \) which only gives one linearly independent eigenvector \( \overline{\mathbf{x}} \).

Then the general solution is

\[
\mathbf{x}(t) = c_1 \overline{\mathbf{x}} e^{rt} + c_2 (\overline{\mathbf{x}} t e^{rt} + \overline{\mathbf{x}} e^{rt}),
\]

where \( \overline{\mathbf{x}} \) satisfies

\[
(A - r I_{2 \times 2}) \overline{\mathbf{x}} = \overline{\mathbf{x}}.
\]