Let's return to the general form of a first order ODE

\[(*) \quad M(x,y) + N(x,y) \cdot y' = 0.\]

If \(M\) is just a function of \(x\) and \(N\) is just a function of \(y\), we call \((*)\) separable and we've learned how to determine all solutions by the method of separation of variables.

Suppose now that we are in the situation that there exists a function \(\phi(x,y)\) such that its partial derivatives satisfy

\[\frac{\partial \phi}{\partial x}(x,y) = M(x,y), \quad \frac{\partial \phi}{\partial y}(x,y) = N(x,y).\]

When we think of \(y\) as an implicit function of \(x\), i.e. \(y = \phi(x)\), we have the following fact from Calc III

\[\frac{d}{dx} \left( \phi(x, \phi(x)) \right) = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{dx} = y'.\]
But then we can conclude that
\[
\frac{d}{dx} \left( \psi(x, y(x)) \right) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \cdot y' = M(x, y(x)) + N(x, y(x)) \cdot y' = 0
\]

Thus, solutions to the ODE \((\ast)\) are implicitly given by
\[
\psi(x, y) = C,
\]
where \(C\) is an arbitrary constant.

\underline{Example: \[p. 95\]}

Consider the ODE
\[
\frac{2x + y^2}{x} + \frac{2xy \cdot y'}{y} = 0.
\]
Here the function
\[
\psi(x, y) = x^2 + xy^2
\]
has the property
\[
\frac{\partial \psi}{\partial x} (x, y) = 2x + y^2 = M(x, y),
\]
\[
\frac{\partial \psi}{\partial y} (x, y) = 2xy = N(x, y).
\]
Thus, the ODE can be written as
\[
\frac{d}{dx} \left( \frac{x^2 + xy^2}{y} \right) = \psi'(x, y) = 0.
\]
Integrating in \( x \) yields that the general solutions to the ODE are implicitly given by
\[
x^2 + xy^2 = C.
\]
Recall that the curve in the \( x-y \) plane of points satisfying an equation like
\[
x^2 + xy^2 = C
\]
will in general not look like the graph of a function.

This raises two key questions:

Q1: How do we know if such a function \( y(x,y) \) exists for \( (x) \)?

Q2: How can we compute \( y(x,y) \)?

**Theorem:**

Let the functions \( M, N, \frac{\partial M}{\partial y}, \) and \( \frac{\partial N}{\partial x} \) be continuous in the rectangular region
\[
R : a < x < b, \ y < y < c.
\]
Then there exists a function \( y(x, y) \) satisfying
\[
\frac{\partial y}{\partial x}(x, y) = M(x, y), \quad \frac{\partial y}{\partial y}(x, y) = N(x, y) \quad \text{in} \ R
\]
if and only if
\[
\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y) \quad \text{in} \ R.
\]

**Proof:** (see textbook)
Definition:

The first order ODE

\[ M(x,y) + N(x,y) \cdot y' = 0 \]

is called exact in a rectangular region \( R: a < x < b, \quad c < y < d \) if

1. \( M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x} \) are continuous on \( R \)
2. \( \frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y) \) for all \((x,y) \in R\).

Let's see how the theorem helps in practice to determine the solutions to an exact ODE.

Example:

Solve the IVP

\[ 2x - y + (2y - x) \cdot y' = 0 \quad , \quad y(1) = 3. \]

Solution:

First we check whether the ODE is exact.

Here we have

\[ M(x,y) = 2x - y, \quad N(x,y) = 2y - x \]

and

\[ \frac{\partial M}{\partial y}(x,y) = -1, \quad \frac{\partial N}{\partial x}(x,y) = -1 \]

\[ \Rightarrow \frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2 \]

Thus, the ODE is exact and since \( M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x} \) are continuous on \( \mathbb{R}^2 \) there exists a function \( y(x,y) \) such that

\[ \frac{\partial y}{\partial x}(x,y) = M(x,y), \quad \frac{\partial y}{\partial y}(x,y) = N(x,y). \]
To actually compute $y(x,y)$ we integrate with respect to $x$ the equation

\[ \frac{2y}{2x}(x,y) = M(x,y) = 2x - y \]

for every fixed $y$ to get

\[ y(x,y) = \int \frac{2y}{2x}(x,y) \, dx = \int M(x,y) \, dx = \int 2x - y \, dx = x^2 - xy + h(y) \]

To determine the function $h(y)$ we use that

\[ \frac{2y}{2y}(x,y) = M(x,y) \]

to infer

\[ 2y - x = M(x,y) = \frac{2y}{2y}(x,y) = \frac{2}{2y} \left( x^2 - xy + h(y) \right) = -x + h'(y) \]

\[ \Rightarrow h'(y) = 2y \Rightarrow h(y) = y^2 + \text{const}. \]

Thus, the general implicit solutions to the ODE are given by

\[ x^2 - xy + y^2 = C. \]
To fulfill the initial value \( y(1) = 3 \) we must have
\[ 1^2 - 1 \cdot 3 + z^2 = C \implies C = 7. \]
Hence, the particular solution to the IVP is implicitly given by
\[ x^2 - xy + y^2 = 7. \]
Here it is in fact possible to solve for \( y \) and determine the interval of existence
\[ y(x) = \frac{x + \sqrt{28-3x^2}}{2}, \quad x \in \left( -\frac{\sqrt{28}}{3}, \frac{\sqrt{28}}{3} \right). \]

**Remark:**
Sometimes it is possible to convert an ODE, which is not exact, into an exact ODE using an integrating factor, see p. 99 in our textbook. We will not pursue this in this course.