Second Order Linear ODEs

A second order ordinary differential equation is of the general form

\[ (*) \quad \frac{d^2 y}{dt^2} = f(t, y, y') \]

where \( f \) is some given function, \( t \) is the independent variable, and \( y \) the dependent variable (i.e. the unknown function).

**Definition:**

We say that the second order ODE \( (*) \) is \underline{linear} if it can be written as

\[ (**) \quad y'' + p(t) \cdot y' + q(t) \cdot y = g(t) \]

or

\[ (***) \quad P(t) \cdot y'' + Q(t) \cdot y' + R(t) \cdot y = G(t). \]

Otherwise we say that \( (*) \) is \underline{nonlinear}.

**Note:**

If \( P(t) \neq 0 \), we can divide \( (***) \) by \( P(t) \) and obtain \( (**) \) with

\[ y(t) = \frac{Q(t)}{P(t)}; \quad q(t) = \frac{R(t)}{P(t)}; \quad g(t) = \frac{G(t)}{P(t)}. \]
An initial value problem (IVP) for a second order ODE \((*)\) comprises two initial values, usually\[
y(\tau_0) = y_0, \quad y'(\tau_0) = y'_0
\]
for some real numbers \(y_0, y'_0\).

**Definition:**

A second order linear ODE is called homogeneous if \(g(t) \equiv 0\) for all \(t\) \((\text{in }(*)\))
(or \(G(t) \equiv 0\) for all \(t\) \((\text{in }(***)\))).

In general, second order ODE are very hard to solve. Even second order linear ODE with non-constant coefficients \(P(t), Q(t), R(t)\) can be hard.

However, for constant coefficients \(P(t) = a, \quad Q(t) = b, \quad R(t) = c\), \(a, b, c \in \mathbb{R}\),
a second order linear ODE can always be solved. We start discussing the homogeneous case:

\[
a \cdot y'' + b \cdot y' + c \cdot y = 0.
\]
Homogeneous second order ODEs with constant coefficients

Let's start with a concrete example

\[ y'' = y \quad \text{(here } a = 1, b = 0, c = -1) \]

If you experiment a bit with elementary functions (polynomials, exponential functions, trigonometric functions, ...) you may find that \( y_1(t) = e^t \) and \( y_2(t) = e^{-t} \) are solutions.

What about constant multiples of \( y_1, y_2 \)? And what about sums of these?

In fact for any constants \( c_1, c_2 \in \mathbb{R} \),

\[ y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t) \]

is also a solution. Let's check this:

\[ y''(t) - y(t) \]

\[ = (c_1 \cdot y''_1(t) + c_2 \cdot y''_2(t)) - (c_1 \cdot y_1(t) + c_2 \cdot y_2(t)) \]

\[ = c_1 \cdot (y''_1(t) - y_1(t)) + c_2 \cdot (y''_2(t) - y_2(t)) \]

\[ = 0 = 0 \]

\[ \therefore \]
What if we have an IVP
\[ y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1. \]
We can hope that all solutions are of the form
\[ y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t) \]
and correspondingly determine \( c_1, c_2 \):

\[ 2 = y(0) = c_1 \cdot y_1(0) + c_2 \cdot y_2(0) = c_1 + c_2 \]
\[ = e^0 = 1 \]
\[ = e^{-0} = 1 \]

\[ -1 = y'(0) = c_1 \cdot y_1'(0) + c_2 \cdot y_2'(0) = c_1 - c_2 \]
\[ = e^0 = 1 \]
\[ = -e^{-0} = -1 \]
Thus, we have to solve
\[ c_1 + c_2 = 2, \]
\[ c_1 - c_2 = -1 \]
\[ \Rightarrow c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}. \]

\[ \Rightarrow \text{The particular solution to the IVP is} \]
\[ y(t) = \frac{1}{2} e^t + \frac{3}{2} e^{-t}. \]
Let's return to the general case of a 2nd order homogeneous and linear ODE with constant coefficients:

\[(x4) \ a \cdot y'' + b \cdot y' + c = 0\]

Assume that it has a solution of exponential type:

\[y(t) = e^{rt}\]

Since \[y'(t) = r \cdot e^{rt}\] and \[y''(t) = r^2 \cdot e^{rt}\], we must have

\[a \cdot r^2 e^{rt} + b \cdot r e^{rt} + c e^{rt} = 0\]

\[\Rightarrow (ar^2 + br + c) \cdot e^{rt} = 0.\]

Since \[e^{rt} \neq 0\] for all \(t\), we obtain

\[(x5) \ a \cdot r^2 + b \cdot r + c = 0.\]

This equation is called the characteristic equation for the ODE \((x4)\).

[The characteristic equation of \(y'' - y = 0\) is \(r^2 - 1 = 0\)]

Thus, \[y(t) = e^{rt}\] is a solution to \((x4)\) if and only if \(r\) is a root of \((x5)\)!

The roots of \((x5)\) are given by

\[r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.\]
There are three cases:
(1) two real and distinct roots \( r_1 \neq r_2 \), \( r_1, r_2 \in \mathbb{R} \)
(2) a repeated real root \( r_1 = r_2 \in \mathbb{R} \)
(3) two complex roots which are conjugates \( (\overline{r_1} = r_2, \ r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}) \)

Here we further discuss the first case (1) of two real and distinct roots \( r_1, r_2 \in \mathbb{R} , r_1 \neq r_2 \). [We discuss the other two cases later.]

Then \( y_1(t) = e^{r_1 t} \) and \( y_2(t) = e^{r_2 t} \) are solutions to (**) .

In fact for any constants \( c_1, c_2 \in \mathbb{R} \),

\[ y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t) \]

is then also a solution to (**). Check this!

\[
\begin{align*}
y'(t) &= c_1 r_1 y_1(t) + c_2 r_2 y_2(t) \\
y''(t) &= c_1 r_1^2 y_1(t) + c_2 r_2^2 y_2(t)
\end{align*}
\]

\[ \Rightarrow a \cdot y'' + b \cdot y' + c = 0 \]

\[ = c_1 \cdot \left( a r_1^2 + b r_1 + c \right) e^{r_1 t} + c_2 \cdot \left( a r_2^2 + b r_2 + c \right) e^{r_2 t} = 0 \]

\[ = 0 \quad \text{if} \ r_1 \text{ is root of char. equation} \]

\[ \text{and} \ r_2 \text{ is root of char. equation} \]

\[ \text{for } x^2 + b x + c = 0 \]
We will show later that when \( r_1, r_2 \) are real and distinct, then the general solution to
\[
a \cdot y'' + b \cdot y' + c = 0
\]
is given by
\[
y(t) = c_1 \cdot e^{r_1 \cdot t} + c_2 \cdot e^{r_2 \cdot t}, \quad c_1, c_2 \in \mathbb{R}.
\]
For the IVP
\[
y(t_0) = y_0, \quad y'(t_0) = y'_0,
\]
we can then determine the particular solution:
\[
y'(t) = c_1 \cdot r_1 \cdot e^{r_1 \cdot t} + c_2 \cdot r_2 \cdot e^{r_2 \cdot t}
\]

\[
(\Rightarrow) \quad y_0 = y(t_0) = c_1 e^{r_1 \cdot t_0} + c_2 e^{r_2 \cdot t_0}
\]

\[
y'_0 = y'(t_0) = c_1 \cdot r_1 \cdot e^{r_1 \cdot t_0} + c_2 \cdot r_2 \cdot e^{r_2 \cdot t_0}
\]

Solving the last two equations for \( c_1, c_2 \) yields
\[
c_1 = \frac{y'_0 - y_0 \cdot r_2}{r_1 - r_2} \cdot e^{-r_1 \cdot t_0}
\]
\[
c_2 = \frac{y_0 \cdot r_1 - y'_0 \cdot r_1}{r_1 - r_2} \cdot e^{-r_2 \cdot t_0}
\]
Example: Find the general solution to

\[ y'' - 4y + 3y = 0 \]

The characteristic equation is

\[ r^2 - 4r + 3 = 0 \]

\[ \Rightarrow (r-1)(r-3) = 0 \Rightarrow r = 1 \text{ and } r = 3 \]

Thus, the general solution is

\[ y(t) = c_1 e^t + c_2 e^{3t} \]