Solutions of 2nd order linear homogeneous ODEs: the Wronskian

Before we consider the other two cases where the characteristic equation has repeated real roots or complex roots, we first discuss general properties of the set of solutions to linear homogeneous second order ODEs

(*) \( y'' + p(t) \cdot y' + q(t) \cdot y = 0 \).

To this end we introduce the operator

\[ L \phi := \phi'' + p(t) \cdot \phi' + q(t). \]

An operator maps functions to functions. Here, the operator \( L \) is defined for all twice differentiable functions \( \phi \) on an open interval \( I = (a, b) \) with \( a \in \mathbb{R} \cup \mathbb{R}^- \)

and \( b \in \mathbb{R} \cup \mathbb{R}^+ \).

We can also write the operator \( L \) as

\[ L := \frac{d^2}{dt^2} + p(t) \cdot \frac{d}{dt} + q(t). \]
Note that a function \( y(t) \) is a solution to the ODE \( \phi \) on an interval \( I \) if and only if

\[
L[y](t) = 0 \quad \text{for all } t \in I.
\]

We have the following fundamental result about the IVP for second order linear ODEs:

**Theorem:**

Let \( p(t) \) and \( q(t) \) be continuous on an open interval \( I \) with \( t_0 \in I \). Then the IVP

\[
y'' + p(t) \cdot y' + q(t) \cdot y = 0,
\]

\[
y(t_0) = y_0, \quad y'(t_0) = y'_0
\]

has a unique solution \( y = \phi(t) \) that is defined on the whole interval \( I \).

**Remarks:**

- The theorem guarantees both existence and uniqueness of the solution to the IVP.

- It also guarantees that the solution \( y = \phi(t) \) is defined throughout the whole interval \( I \) where \( p(t), q(t), q(t) \) are continuous.
Let's continue with the operator \( L \):

We say that the operator \( L[\phi] \) is linear if for any two functions \( \phi_1, \phi_2 \) and real numbers \( c_1, c_2 \in \mathbb{R} \):

\[
L \left[ c_1 \cdot \phi_1 + c_2 \cdot \phi_2 \right] = c_1 \cdot L[\phi_1] + c_2 \cdot L[\phi_2].
\]

This is indeed the case for \( L = \frac{d^2}{dt^2} + \mu(t) \frac{d}{dt} + q(t) \):

**Proof:**

\[
L \left[ c_1 \cdot \phi_1 + c_2 \cdot \phi_2 \right]
= \frac{d^2}{dt^2} \left[ c_1 \cdot \phi_1(t) + c_2 \cdot \phi_2(t) \right] + \mu(t) \cdot \frac{d}{dt} \left[ c_1 \cdot \phi_1(t) + c_2 \cdot \phi_2(t) \right] + q(t) \cdot (c_1 \cdot \phi_1(t) + c_2 \cdot \phi_2(t))
= c_1 \cdot \phi_1''(t) + c_2 \cdot \phi_2''(t) + \mu(t) \left( c_1 \cdot \phi_1'(t) + c_2 \cdot \phi_2'(t) \right) + q(t) \cdot (c_1 \cdot \phi_1(t) + c_2 \cdot \phi_2(t))
= c_1 \cdot \left( \phi_1''(t) + \mu(t) \cdot \phi_1'(t) + q(t) \cdot \phi_1(t) \right) + c_2 \cdot \left( \phi_2''(t) + \mu(t) \cdot \phi_2'(t) + q(t) \cdot \phi_2(t) \right)
= c_1 \cdot L[\phi_1](t) + c_2 \cdot L[\phi_2](t).
\]

The term \( c_1 \cdot \phi_1(t) + c_2 \cdot \phi_2(t) \) is called a linear combination of the functions \( \phi_1(t), \phi_2(t) \).
Q: If $y_1(t), y_2(t)$ are solutions to the linear homogeneous second-order ODE
\[ y'' + p(t)\cdot y' + q(t)\cdot y = 0, \]
what about their linear combinations?

Theorem: (Principle of Superposition)

If $y_1$ and $y_2$ are two solutions to
\[ L[y] = y'' + p(t)\cdot y' + q(t)\cdot y = 0, \]
then the linear combination $c_1\cdot y_1 + c_2\cdot y_2$ is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Proof:

Since the operator $L$ is linear, we have
\[ L\left[ c_1\cdot y_1 + c_2\cdot y_2 \right] = c_1\cdot \left[ L[y_1] \right] + c_2\cdot \left[ L[y_2] \right] = 0. \]
Thus, $c_1\cdot y_1 + c_2\cdot y_2$ is also a solution.
Now suppose that we have found two solutions \( y_1(t) \) and \( y_2(t) \) to (*) i.e. to

\[
y'' + p(t)y' + q(t) = 0.
\]

Q: Are all solutions to (*) a linear combination \( c_1y_1(t) + c_2y_2(t) \)?

In other words, for any initial values \( y_0, y'_0 \), is the unique solution to the IVP for (*) with \( y(t_0) = y_0, y'(t_0) = y'_0 \) of the form \( y(t) = c_1y_1(t) + c_2y_2(t) \) for some choice of \( c_1, c_2 \in \mathbb{R} \)?

Given the initial values \( y(t_0) = y_0, y'(t_0) = y'_0 \), let's try to "solve" the IVP with the ansatz

\[
y(t) = c_1y_1(t) + c_2y_2(t).
\]

We have

\[
y'(t) = c_1y'_1(t) + c_2y'_2(t)
\]

and we must achieve that

\[
c_1y_1(t_0) + c_2y_2(t_0) = y_0 \quad (= y(t_0)) \quad \text{(**)}
\]

\[
c_1y_1'(t_0) + c_2y_2'(t_0) = y'_0 \quad (= y'(t_0)).
\]
This is a system of two equations that we have to solve for the two unknowns \( c_1, c_2 \).

Three cases are possible:

(i) no solution (lines parallel, but not identical)

(ii) one solution (lines cross)

(iii) infinitely many solutions (lines identical)

We obtain \[ c_1 = \frac{Y_0 \cdot Y_2'(t_0) - Y_0' \cdot Y_2(t_0)}{Y_1(t_0) \cdot Y_2'(t_0) - Y_1'(t_0) \cdot Y_2(t_0)}, \]

\[ c_2 = \frac{-Y_0 \cdot Y_1'(t_0) + Y_0' \cdot Y_1(t_0)}{Y_1(t_0) \cdot Y_2'(t_0) - Y_1'(t_0) \cdot Y_2(t_0)}. \]

Note that in the expressions for \( c_1 \) and \( c_2 \), the denominator is the same (but the numerators are different).

The above three cases correspond to:

(i) only denominator is zero

(ii) denominator is non-zero

(iii) numerator and denominator are zero
If we write the system of equations \((xy)\) as a matrix equation

\[
\begin{bmatrix}
Y_1(t_0) & Y_2(t_0) \\
Y_1'(t_0) & Y_2'(t_0)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
Y_0 \\
Y_0'
\end{bmatrix},
\]

then the denominator in the expressions for \(c_1, c_2\) is the determinant of the matrix

\[
Y_1(t_0) \cdot Y_2'(t_0) - Y_1'(t_0) \cdot Y_2(t_0) = \begin{vmatrix}
Y_1(t_0) & Y_2(t_0) \\
Y_1'(t_0) & Y_2'(t_0)
\end{vmatrix}
\]

We call

\[
W = W(y_1, y_2)(t_0) = \begin{vmatrix}
Y_1(t_0) & Y_2(t_0) \\
Y_1'(t_0) & Y_2'(t_0)
\end{vmatrix}
\]

the \underline{Wronskian (determinant)} of the solutions \(y_1\) and \(y_2\) \((at t_0)\).

Thus, if the Wronskian \(W(y_1, y_2)(t_0) \neq 0\) is non-zero, then the equations \((xy)\) have a unique solution \((c_1, c_2)\).