At the end of our last lecture we arrived at the following conclusion:

**Theorem:**

Suppose that \( y_1(t) \) and \( y_2(t) \) are two solutions to

\[
L[y] = y'' + p(t)y' + q(t)y = 0.
\]

Suppose also that the initial values

\[
y(t_0) = y_0, \quad y'(t_0) = y_0'
\]

are assigned and satisfy

\[
W(y_1, y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0.
\]

Then there exist \( c_1, c_2 \in \mathbb{R} \) s.t.

\[
y(t) = c_1y_1(t) + c_2y_2(t)
\]

solves this IVP.

**Example:**

We found that

\[
y'' - 4y + 3y = 0
\]

has the solutions \( y_1(t) = e^t \) and \( y_2(t) = e^{3t} \).

By considering the roots of the characteristic equation

\[
r^2 - 4r + 3 = 0
\]

Their Wronskian at any time \( t \) is

\[
W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^t & e^{3t} \\ e^t & 3e^{3t} \end{vmatrix} = e^t \cdot 3e^{3t} - e^t \cdot e^{3t} = 2e^{4t} \neq 0 \quad (\text{for all } t \in \mathbb{R}).
\]
The Wronskian therefore has the following significance for the structure of the set of all solutions to $L[y] = y'' + p(t)y' + q(t)y = 0$.

**Theorem:**

Suppose that $y_1(t)$, $y_2(t)$ are two solutions to 

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$ 

If there exists a time $t_0$ with $W(y_1, y_2)(t_0) \neq 0$, then the family of solutions 

$$y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$$ 

with arbitrary $c_1, c_2 \in \mathbb{R}$ includes every solution to $L[y] = 0$.

**Proof:**

Let $\phi(t)$ be any solution to $L[y] = 0$.

Since $W(y_1, y_2)(t_0) \neq 0$, the IVP 

$$L[y] = 0, \quad y(t_0) = \phi(t_0), \quad y'(t_0) = \phi'(t_0)$$

has a solution $y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ for some choice of $c_1, c_2 \in \mathbb{R}$ by the previous theorem.

But then, by uniqueness [of solutions to the IVP, from the last lecture], we must have 

$$\phi(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t).$$
Thus, for a given 2nd order linear homogeneous ODE \( \text{L}y(t) = 0 \), if you find two solutions \( y_1(t) \) and \( y_2(t) \) with non-zero Wronskian \( W(y_1, y_2)(t) \neq 0 \) at some time \( t_0 \), then all solutions to \( \text{L}y(t) = 0 \) are of the form

\[
y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t),
\]

called the general solution or the fundamental set of solutions to \( \text{L}y(t) = 0 \).

**Example:**

Let \( y_1(t) = e^{r_1 t} \) and \( y_2(t) = e^{r_2 t} \) be two solutions to \( \text{L}y(t) = y'' + p(t)y' + q(t)y = 0 \). Then their Wronskian at any time \( t \) is

\[
W(y_1, y_2)(t) = \begin{vmatrix}
y_1(t) & y_2(t) \\
y_1'(t) & y_2'(t)
\end{vmatrix} = e^{r_1 t} e^{r_2 t} - e^{r_1 t} e^{r_2 t} = (r_2 - r_1) e^{(r_1+r_2)t}.
\]

Thus, if \( r_1 \neq r_2 \), then

\[
y(t) = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}
\]
is a fundamental set of solutions.
Example:
\[ y_1(t) = e^t \sin(t), \quad y_2(t) = e^t \cos(t) \]

\[ \Rightarrow y_1'(t) = e^t \sin(t) + e^t \cos(t), \quad y_2'(t) = e^t \cos(t) - e^t \sin(t) \]

\[ \Rightarrow W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t) \]

\[ = e^t \sin(t) \left( e^t \cos(t) - e^t \sin(t) \right) \]

\[ - \left( e^t \sin(t) + e^t \cos(t) \right) \cdot e^t \cos(t) \]

\[ = -e^{2t} \sin^2(t) - e^{2t} \cos^2(t) \]

\[ = -e^{2t} = 0 \quad \text{for all} \ t \in \mathbb{R}. \quad \]

The next theorem gives an explicit formula for the Wronskian with important consequences:

Theorem:
Let \( y_1(t) \) and \( y_2(t) \) be two solutions to
\[ L[y] = y'' + p(t) y' + q(t) y = 0, \]
where \( p(t) \) and \( q(t) \) are continuous on an open interval I. Then the Wronskian \( W(y_1, y_2)(t) \) is given by

\[ W(y_1, y_2)(t) = C \cdot \exp \left( -\int p(t) \, dt \right), \]

where \( C \) is a constant that depends on \( y_1 \) and \( y_2 \), but not on \( t \).
Conclusion:

\( W(x_1, x_2)(t) \) is either zero for all \( t \in [a,b] \) (if \( c = 0 \)) or else is never zero in \( I \) (if \( c \neq 0 \)).

Proof of Theorem:

We know that \( x_1 \) and \( x_2 \) are solutions to \( L[y] = 0 \):

1. \( y_1'' + p(t) \cdot y_1' + q(t) \cdot y_1 = 0 \)
2. \( y_2'' + p(t) \cdot y_2' + q(t) \cdot y_2 = 0 \)

Now multiply (1) by \(-y_2\), multiply (2) by \(y_1\), and add the resulting equations to get:

3. \( (y_1'y_2'' - y_1''y_2') + p(t) \cdot (y_1y_2' - y_1'y_2) = 0 \).

Next, let

\[ W(t) := W(x_1, x_2)(t) = y_1(t) \cdot y_2'(t) - y_1'(t) \cdot y_2(t) \]

and observe that [check!]

\[ W'(t) = y_1(t) \cdot y_2''(t) - y_1''(t) \cdot y_2(t) \]

Then we can rewrite (3) as:

\[ W'(t) + p(t) \cdot W(t) = 0. \]

But this is a linear first order ODE for \( W(t) \) which we can solve immediately [with the method of integrating factors] to obtain:

\[ W(t) = C \cdot \exp \left( - \int p(t) \, dt \right), \]

where \( C \) is a constant (that depends on \( x_1 \) and \( x_2 \)).