Section 3.3: Complex roots of the characteristic equation

Let's return to the case of 2nd order linear, homogeneous ODEs with constant coefficients

\[ a \cdot y'' + b \cdot y' + c \cdot y = 0. \]

The associated characteristic equation is

\[ a \cdot r^2 + b \cdot r + c = 0 \]

and has roots

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

If \( b^2 - 4ac > 0 \), we obtain two distinct real roots; this case has been discussed in the last lectures.

Now suppose \( b^2 - 4ac < 0 \), then we obtain two complex roots which are complex conjugates of each other:

\[ r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu \]

with \( \lambda, \mu \in \mathbb{R} \) and \( \mu \neq 0 \).

Check that

\[ \lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{-(b^2 - 4ac)}}{2a}. \]
Then the two functions

\[ Y_1(t) = e^{(\lambda + i\mu)t} \quad \text{and} \quad Y_2(t) = e^{(\lambda - i\mu)t} \]

are solutions to the ODE (a).

Q: What does this mean? What is meant by the expressions \( e^{(\lambda + i\mu)t} \)?

Euler's formula states that

\[ e^{ia} = \cos(a) + i\sin(a) \quad \text{for} \quad a \in \mathbb{R} \]

and correspondingly

\[ e^{(\lambda + i\mu)t} = e^{\lambda t}e^{i\mu t} = e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \]

Thus,

\[ Y_1(t) = e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \]
\[ Y_2(t) = e^{\lambda t}(\cos(\mu t) - i\sin(\mu t)) \]

However, these are not real-valued solutions to (a)!

For an ODE (a) with real-valued coefficients, it would be nice to have real-valued solutions.


Use the superposition principle:
Any linear combinations of $y_1(t)$ and $y_2(t)$ are also solutions to $(*)$ and thus in particular
\[
\frac{1}{2} \left( y_1(t) + y_2(t) \right) = \frac{1}{2} \left( e^{xt} \cos(\mu t) + i \sin(\mu t) \right) + e^{xt} \left( \cos(\mu t) - i \sin(\mu t) \right) = e^{xt} \cos(\mu t)
\]
and
\[
\frac{1}{2i} \left( y_1(t) - y_2(t) \right) = \frac{1}{2i} \left( e^{xt} \cos(\mu t) + i \sin(\mu t) \right) - e^{xt} \left( \cos(\mu t) - i \sin(\mu t) \right) = e^{xt} \sin(\mu t)
\]
are solutions to $(*)$, which are real-valued!
(In fact, these are the real and imaginary parts of the solutions $y_1(t)$ and $y_2(t).$)

Let's denote them by
\[
u(t) = e^{xt} \cos(\mu t), \quad v(t) = e^{xt} \sin(\mu t).
\]
Then their Wronskian is
\[
W(u,v)(t) = \mu \cdot e^{2xt},
\]
which is non-zero for $\mu \neq 0$.

To check this computation!
Hence, when the characteristic equation
\[ a \cdot r^2 + b \cdot r + c = 0 \]
(associated with the ODE \( a \cdot y'' + b \cdot y' + c \cdot y = 0 \))
has complex roots \( \lambda \pm \mu i \) with \( \mu \neq 0 \),
then the general solution of
\[ a \cdot y'' + b \cdot y' + c \cdot y = 0 \]
is given by
\[ y(t) = c_1 \cdot e^{\lambda t} \cos(\mu t) + c_2 \cdot e^{\lambda t} \sin(\mu t) \]
\[ = e^{\lambda t} \left( c_1 \cos(\mu t) + c_2 \sin(\mu t) \right) \]
for arbitrary \( c_1, c_2 \in \mathbb{R} \).

Example: Solve the IVP
\[ y'' - 2y' + 5y = 0 \quad \text{, \quad with} \quad y(0) = 2, \quad y'(0) = 1 \]

Solution:
The characteristic equation is
\[ r^2 - 2r + 5 = 0 \]
and has roots
\[ r_{1,2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i. \]
Hence, the general solution to the ODE is
\[ y(t) = e^t \cdot (c_1 \cos(2t) + c_2 \sin(2t)) \]
for \( c_1, c_2 \in \mathbb{R} \) arbitrary.

Then
\[
y'(t) = e^t \left( c_1 \cos(2t) + c_2 \sin(2t) \right) \\
+ e^t \left( -2c_1 \sin(2t) + 2c_2 \cos(2t) \right).
\]

We must have
\[
z = y(0) = c_1 \\
z = y'(0) = c_1 + 2c_2
\]

\( \Rightarrow \) \( c_1 = z \), \( c_2 = \frac{1}{2} z \)

The solution to the IVP therefore is
\[ y(t) = e^t \cdot \left( 2 \cos(2t) + \frac{1}{2} \sin(2t) \right). \]