Section 2.4: Linear vs. Nonlinear ODEs

Last week we discussed two methods to determine the solutions to certain types of first order ODEs:

- **method of integrating factors** works for all first order linear ODEs
  \[ \frac{dx}{dt} + p(t) \cdot y = q(t). \]

- **method of separation of variables** works for all first order separable ODEs
  \[ M(x) + N(y) \cdot \frac{dy}{dx} = 0. \]

All examples of IVPs discussed so far had a unique solution.

Q: Does every IVP have a unique solution?

For linear first order ODEs the following theorem gives a very satisfactory answer:

**Theorem:**

Let \( p(t), q(t) \) be continuous on an open interval \( I = (a, b) \) with \( t_0 \in I \). Then for any prescribed value \( y_0 \in \mathbb{R} \) there exists a unique function \( y = \phi(t) \) that satisfies

\[ y'(t) + p(t) \cdot y(t) = q(t) \]

for each \( t \in I \) and that also satisfies the initial condition

\[ y(t_0) = y_0. \]
Proof: follows from a close inspection of our derivation of the method of integrating factors.

Remarks:

(a) The theorem asserts both the existence and the uniqueness of the solution to the IVP.

(b) It also states that the solution exists throughout any interval $I \ni t_0$ in which the coefficients $p(t)$ and $q(t)$ are continuous!

→ For nonlinear ODEs, the situation becomes less satisfactory (or more interesting ...) as the following examples illustrate:

Example: (Non-uniqueness of solutions)

Consider the IVP

\[(*) \quad y' = \frac{\pi}{2} \cdot \frac{y^3}{2} \quad \text{for} \quad t > 0, \quad y(0) = 0.\]

It has the solution

$y_1(t) = 0$. But also the function

$y_2(t) = t^{\frac{3}{2}}$

covers the IVP $(*)$

$y_2(t) = \frac{3}{2} \cdot t^2 = \frac{3}{2} \cdot \left(t^{\frac{3}{2}}\right)^{\frac{2}{2}} = \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

as well as the function

$y_3(t) = -t^{\frac{3}{2}}$.\]
In fact, for any $c > 0$, the function
\[ y(t) = \begin{cases} 
0 & 0 \leq t < c \\
\pm (t-c)^{1/2} & t > c
\end{cases} \]
is a solution to the IVP (1).

Thus, the IVP (1) has infinitely many solutions!

**Example:** (The interval of existence of a solution can be difficult to predict)

Consider the IVP
\[ y' = y^2, \quad y(0) = y_0 \text{ for some } y_0 > 0. \]

This is a separable equation and thus we can determine the solution by the method of separation of variables
\[ \frac{1}{y^2} \frac{dy}{dt} = 1 \]
\[ \Rightarrow \int \frac{1}{y^2} \, dy = \int 1 \, dt \]
\[
\Rightarrow -\frac{1}{y} = t + C \\
\Rightarrow y = \frac{-1}{t + C}
\]

To have \( y(0) = y_0 \) we must find the constant \( C \) as follows:

\[
y_0 = y(0) = \frac{-1}{0 + C} \Rightarrow C = -\frac{1}{y_0}
\]

\[
\Rightarrow \text{The solution is the function} \\
y(t) = \frac{-1}{t - \frac{1}{y_0}} = \frac{y_0}{1 - y_0 \cdot t}
\]

with interval of existence \((-\infty, \frac{1}{y_0})\).

- For (nonlinear) first order ODE we have the following theorem guaranteeing the existence and uniqueness of solutions:

**Theorem:**

Let \( f(t, y) \) and \( \frac{\partial f}{\partial y} (t, y) \) be continuous in some rectangle \( \mathcal{R} \) containing \((t_0, y_0)\). Then in some interval \( t_0 - h < t < t_0 + h \) inside \( \mathcal{R} \), there exists a unique solution \( y(t) \) to

\[
(yx) \frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.
\]

**Proof:**

will be discussed later.
Remarks:

(a) Important geometric consequence: Under the assumptions of the theorem, the graphs of two solutions cannot intersect each other!

(b) Existence (without uniqueness) of a solution to the IVP (**) can be established using just the continuity of $f(t, x)$.

(c) The maximal interval where a solution to the IVP (**) exists can in general be difficult to determine.
(d) Let’s revisit the example
\[
\frac{dy}{dt} = \frac{3}{2} \, y^{1/3}; \quad y(t_0) = y_0.
\]
Here \( f(t, y) = \frac{3}{2} \, y^{1/3} \) is defined and continuous for all \( t \in \mathbb{R} \) and \( y \in \mathbb{R} \).

Thus, by Remark (5) solutions are guaranteed to exist.

However, \( \frac{\partial f}{\partial y}(t, y) = \frac{1}{2} \, y^{-2/3} \) is not continuous along the \( y = 0 \) line (in the \( t-y \) plane).

For starting values \( y(t_0) = 0 \) solutions exist but they may not be unique!

(For all other starting values \( y_0 \neq 0 \) they are unique!)