Introduction

What is a differential equation?

Let \( y = f(x) \) be an unknown function between two variables:
- \( x \): independent variable
- \( y \): dependent variable

Consider the differential equation
\[
(\star) \quad \frac{d^2y}{dx^2} = -4y \quad \text{(or write as } y'' = -4y) \]

We can check that the functions
\[
y_1(x) = \sin(2x) \quad \text{and} \quad y_2(x) = \cos(2x)
\]
are solutions to (\star).

Compare with a (traditional) algebraic equation
\[
x^2 - 3x + 2 = 0
\]
\[
\Rightarrow (x-1)(x-2) = 0,
\]
whose solutions are numbers (namely \( x_1 = 1, x_2 = 2 \)).

More examples of differential equations:

(a) \( \frac{dy}{dx} + ay = 0 \), \( a \in \mathbb{R} \)

(b) Newton’s 2nd law of motion:
\[
F = ma
\]
force \ mass \  acceleration: \( a = \frac{d^2x}{dt^2} \)
Consider a mass $m$ attached to a stretched spring: $F = -k \cdot y$, $k > 0$

Newton’s 2nd law $\ddot{y} = m \cdot \frac{d^2y}{dt^2}$ or $\frac{d^2y}{dt^2} = -\frac{k}{m} \cdot y$

which is of the form $(*)$

(c) Oscillating pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \cdot \sin(\theta) = 0$$

A differential equation that describes some process in nature is called a mathematical model.

Many principles in physics, chemistry, engineering... are formulated in terms of differential equations.
Definition:

An ordinary differential equation (ODE) is an equation involving an unknown function of one independent variable and some of its derivatives.

→ "ordinary" means that the unknown function is just a function of one independent variable.

→ Partial differential equations are equations involving an unknown function of more than one independent variables and some of its (partial) derivatives.

Examples of partial differential equations (PDEs):

- Wave equation: \((\text{for } u(t, x_1, x_2, x_3))\)
  \[- \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0\]
  → Maxwell's equations of electromagnetism in vacuum lead to wave equations.

- Schrödinger equation:
  \[
i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0\]
  → Governs quantum mechanics

→ This course is about ODEs.
**Definition:**

The order of a differential equation is the order of the highest derivative that appears in the equation.

- (a) is first order ODE,
- (b) and (c) are second order ODEs.

**Definition:**

The general form of an $n$-th order ODE is

\[(**) \quad F(t, y, y', \ldots, y^{(n)}) = 0 \]

$F$ is some expression in $t, y, y', \ldots, y^{(n)}$.

$y$ is the dependent variable $y = y(t)$.

$y^{(i)}$ is the $i$-th derivative of $y$.

Sometimes one can solve an ODE for the highest derivative,

\[ y^{(n)} = g(t, y, y', \ldots, y^{(n-1)}) \]

But this is not always possible, consider for instance

\[ y' + \exp(y'') = y'' \]
Definition:

An $n$-th order ODE is linear if it can be written as

$$a_n(t) \cdot y^{(n)} + a_{n-1}(t) \cdot y^{(n-1)} + \ldots + a_1(t) \cdot y = g(t).$$

Examples:

(i) (a) and (b) are linear, (c) is not.

(ii) $e^t \cdot y^{(3)} + \sin(t) \cdot y' = \cos(t) \rightarrow$ linear

(iii) $y^2 + 1 = y'$ → not linear

Note:

Suppose $y' = f(t, y)$ is a 1st order linear ODE.

Then there must exist functions $\nu(t)$, $g(t)$ such that

$$\dot{y}(t, y) = -\nu(t) \cdot y + g(t)$$

and the ODE can be written in the form

$$y' + \nu(t) \cdot y = g(t).$$

This form will be very important in the next weeks to study 1st order linear ODEs.
Definition:

A solution of the ODE (***) on the interval $I = (a,b) \subset \mathbb{R}$ is a function $y(t)$ that satisfies (***)..

Important questions about an ODE:

- Existence of solutions?
- Uniqueness of solutions?
- Obtaining exact solutions (in terms of elementary functions) and approximating solutions?

Next time:

The ODE is 

$$\frac{dv}{dt} = 10 - \frac{v}{5}$$

is solved by $v(t) = 50 + C \cdot e^{-\frac{t}{5}}$

for all $C \in \mathbb{R}$. [Check it!]

How? What's the role of the parameter $C$?