Section 4.2: Homogeneous equations with constant coefficients

We now discuss how to determine the solutions to \( n \)-th order linear ODEs with constant coefficients.

\((*) L[y] = a_0 \cdot y^{(n)} + a_1 \cdot y^{(n-1)} + \ldots + a_{n-1} \cdot y' + a_n \cdot y = 0\)

where \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) with \( a_0 \neq 0 \).

We use the same idea as for \( 2 \)-nd order linear ODEs, namely that exponential functions \( y = e^{rt} \) are good candidates for solutions to \( L[y] = 0 \).

Assuming that \( L[e^{rt}] = 0 \), we arrive at the characteristic equation

\((***) a_0 \cdot r^n + a_1 \cdot r^{n-1} + \ldots + a_{n-1} \cdot r + a_n = 0 \).

As before, roots of the characteristic equation \((***)\) correspond to solutions \( y(t) = e^{rt} \) to \( L[y] = 0 \).

It is an important (and deeper) result that any polynomial of degree \( n \) has \( n \) zeros \( r_1, r_2, \ldots, r_n \), some of which may be equal and some of which may be complex numbers.
If the characteristic equation has complex roots, then they must occur in conjugate pairs, \( \lambda \pm i\mu \), (because \( a_0, a_1, \ldots, a_n \in \mathbb{R} \)).

Using this fact, the theory we developed for 2nd order linear ODEs carries over to construct a fundamental set of solutions (consisting of \( n \) linearly independent solutions).

\[ y(t) = c_1 e^{\lambda t} + c_2 e^{\mu t} + \cdots + c_n e^{\nu t}. \]

Example: (Complex roots)

If the characteristic equation has complex roots, they must occur in conjugate pairs, \( \lambda \pm i\mu \).

Then \( e^{(\lambda + i\mu) t} \) and \( e^{(\lambda - i\mu) t} \) are complex-valued solutions, which (as for 2nd order ODEs) can be replaced by real-valued solutions

\[ e^{\lambda t} \cos(\mu t) \quad \text{and} \quad e^{\lambda t} \sin(\mu t). \]
The 4th order ODE \( y^{(4)} - y = 0 \) has the characteristic equation

\[ r^4 - 1 = 0 \]

\[ \Rightarrow (r^2 - 1)(r^2 + 1) = 0 \]

\[ \Rightarrow (r+1)(r-1)(r^2+1) = 0, \]

which has roots

\[ r_1 = -1, \quad r_2 = +1, \quad r_3 = -i, \quad r_4 = +i. \]

Thus, the general solution is

\[ y(t) = c_1 e^{-t} + c_2 e^t + c_3 \cos(t) + c_4 \sin(t). \]

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**Example:** (Repeated roots)

Recall that if the 2nd order linear ODE \( ay'' + by' + cy = 0 \) has a repeated real root \( r_1 \), then \( e^{rt} \) and \( t \cdot e^{rt} \) are linearly independent solutions. The same applies for higher order linear ODEs.

Assume that the characteristic equation of a 6th order linear ODE is given by

\[ (r-3)^2 \cdot (r^2-2r+5)^2 = 0. \]

Its roots are

\[ r_1 = r_2 = 3, \quad r_3 = r_4 = 1+2i, \quad r_5 = r_6 = 1-2i. \]
Then the general solution is
\[ y(t) = c_1 e^{3t} + c_2 e^{3t} + c_3 e^t \cos(t) + c_4 e^t \sin(t) \]
\[ + c_5 t e^t \cos(t) + c_6 t e^t \sin(t). \]

Finding particular solutions to

nonhomogeneous higher order linear ODEs

\( \quad \)

Method of undetermined coefficients

The same idea for 2\(^{nd}\) order linear ODEs carries over to higher order linear ODEs:

In the constant coefficient case

\[ a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = g(t) \]

with \( a_0, a_1, \ldots, a_n \), if the nonhomogeneous term \( g(t) \) is a sum of products of polynomials, exponentials and sines and cosines, then one can find a particular solution by making an ansatz of the same type as \( g(t) \) with unknown coefficients.
The coefficients are then determined by substituting this ansatz into the equation.

**Method of variation of parameters**

Let $\xi_{1}(t), \ldots, \xi_{n}(t)$ be a fundamental set of solutions to

$$L[y] = y^{(n)} + p_{1}(t) \cdot y^{(n-1)} + p_{2}(t) \cdot y^{(n-2)} + \ldots + p_{n}(t) y = 0.$$  

To obtain a particular solution $Y(t)$ to $L[y] = g(t)$ we make the ansatz

$$Y(t) = u_{1}(t) \cdot \xi_{1}(t) + u_{2}(t) \cdot \xi_{2}(t) + \ldots + u_{n}(t) \cdot \xi_{n}(t),$$

with unknown functions $u_{1}(t), \ldots, u_{n}(t)$.

To determine these we plug the ansatz for $Y(t)$ into $L[y] = g(t)$ and upon making the additional assumptions

$$u_{1}^{(j)} \cdot \xi_{1}^{(j)} + \ldots + u_{n}^{(j)} \cdot \xi_{n}^{(j)} = 0$$

for $j = 0, 1, \ldots, n-2$, we obtain $n$ equations involving $u_{1}, \ldots, u_{n}$ which we can then solve for $u_{1}, \ldots, u_{n}$, see p. 241-242. Upon integration, we obtain formulas for $u_{1}(t), \ldots, u_{n}(t)$.  

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