Section 3.6: Method of variation of parameters

In the last lecture we discussed the method of undetermined coefficients to find a particular solution to a non-homogeneous 2nd order linear ODE

\( (\dagger) \quad L[y] = y'' + p(t) y' + q(t) y = g(t). \)

This method is however restricted to the constant coefficient case and the non-homogeneous part \( g(t) \) must be a sum of products of exponentials, sines and cosines, and polynomials.

We now introduce a general method to determine a particular solution to (\dagger), called the method of variation of parameters. This works for any non-homogeneous 2nd order linear ODE! We will however have to discuss the applicability of the variations of parameter method in practice.
Let $\mathcal{L}y_1, y_2 \mathcal{L}$ be a fundamental set of solutions to the homogeneous equation $L[y] = 0$, i.e. $c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ is the general solution to $L[y] = 0$.

Now we assume that a particular solution to the non-homogeneous equation $L[y] = g(t)$ is of the form

$$Y(t) = u_1(t) \cdot y_1(t) + u_2(t) \cdot y_2(t)$$

where the unknown coefficients $u_1(t)$ and $u_2(t)$ are now $t$-dependent.

We make the additional assumption that these unknown coefficients satisfy

$$u_1'(t) \cdot y_1(t) + u_2'(t) \cdot y_2(t) = 0.$$  

(As we will see why this is helpful soon...)

To determine the unknown coefficients $u_1(t)$ and $u_2(t)$, we plug in the ansatz (***) for $Y(t)$ into (*) and try to solve for $u_1(t)$ and $u_2(t)$ (making use of (****) along the way):
We have

\[ Y'(t) = \frac{u_1 \cdot Y_1 + u_2 \cdot Y_2}{u_1 \cdot Y_1' + u_2 \cdot Y_2'} \]

\[ = 0 \quad \therefore \quad \text{thanks to \textit{(***)}} \]

\[ \Rightarrow \quad Y'(t) = u_1 \cdot Y_1' + u_2 \cdot Y_2' \]

\[ \Rightarrow \quad Y''(t) = u_1 \cdot Y_1'' + u_1 \cdot Y_1' + u_2 \cdot Y_2'' + u_2 \cdot Y_2' \]

Plugging \( Y(t), Y'(t), Y''(t) \) into \( L[Y] = g(t) \):

\[ \begin{aligned}
( u_1 \cdot Y_1' + u_1 \cdot Y_1'' + u_2 \cdot Y_2' + u_2 \cdot Y_2' ) + p(t) \cdot ( u_1 \cdot Y_1' + u_2 \cdot Y_2' ) \\
+ q(t) \cdot ( u_1 \cdot Y_1 + u_2 \cdot Y_2 ) = g(t)
\end{aligned} \]

\[ \text{Note that the assumption \textit{(***)} ensures that we do not pick up second order derivatives of } u_1 \text{ or } u_2! \]

Rearranging yields

\[ \begin{aligned}
&u_1 \cdot Y_1' + u_2 \cdot Y_2' + u_1 \cdot (Y_1'' + p(t) \cdot Y_1' + q(t) \cdot Y_1) \\
&= 0 \quad \therefore \quad \text{since } Y_1, Y_2 \text{ are solutions to } L[Y] = 0 \\
&+ u_2 \cdot (Y_2'' + p(t) \cdot Y_2' + q(t) \cdot Y_2) = g(t) \\
&= 0 \quad \therefore \quad L[Y] = 0, \text{ for solutions} \\
&\Rightarrow \quad u_1 \cdot Y_1' + u_2 \cdot Y_2' = g(t). 
\end{aligned} \]
Thus, the unknown coefficients must satisfy

\[(1) \ u_1'(t) \cdot y_1(t) + u_2'(t) \cdot y_2(t) = 0\]

\[(2) \ u_1'(t) \cdot y_1'(t) + u_2'(t) \cdot y_2'(t) = g(t).\]

We can solve this system for \(u_1'(t)\) and \(u_2'(t)\):

Multiply \((1)\) by \(y_2(t)\) and multiply \((2)\) by \(-y_2(t)\), then add the resulting equations:

\[\begin{align*}
  u_1'(t) \cdot y_2(t) \cdot y_1(t) - u_2'(t) \cdot y_1'(t) \cdot y_2(t) &= -y_2(t) \cdot g(t) \\
  \Rightarrow \ u_1'(t) \cdot (y_1(t) \cdot y_2'(t) - y_1'(t) \cdot y_2(t)) &= -y_2(t) \cdot g(t) \\
  \Rightarrow \ u_1'(t) &= -\frac{y_2(t) \cdot g(t)}{W(y_1, y_2)(t)} \quad \text{since \( W(y_1, y_2)(t) \neq 0 \) by \(W(y_1, y_2)(t)\) is a fundamental set of solutions to \(L[y] = 0\)} \\
  \Rightarrow \ u_2'(t) &= -\frac{y_1(t)}{y_2(t)} \cdot u_1'(t) = \frac{y_1(t) \cdot g(t)}{W(y_1, y_2)(t)}
\end{align*}\]

Thus, after integrating in \(t\), we find that

\[\begin{align*}
  u_1(t) &= -\int \frac{y_2(t) \cdot g(t)}{W(y_1, y_2)(t)} \, dt \\
  u_2(t) &= +\int \frac{y_1(t) \cdot g(t)}{W(y_1, y_2)(t)} \, dt.
\end{align*}\]
Then
\[ Y(t) = u_1(t) \cdot y_1(t) + u_2(t) \cdot y_2(t) \]
is a particular solution to \( L[y] = g(t) \),
and the general solution to \( L[y] = g(t) \)
is given by
\[ y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t) + Y(t). \]

**Example:**
Find the general solution to
\[ y'' + 4y = \frac{3}{\sin(2t)}, \quad 0 < t < \frac{\pi}{2}. \]

**Solution:**
We first determine a fundamental set of solutions for the associated homogeneous equation
\[ y'' + 4y = 0. \]
Its characteristic equation
\[ r^2 + 4 = 0 \quad \Rightarrow \quad r^2 = -4 \]
has roots \( r_1, r_2 = \pm 2i \).
Thus, \( \{\sin(2t), \cos(2t)\} \) is a fundamental set of solutions to \( y'' + 4y = 0 \).

Now we use the method of variation of parameters to find one particular solution \( Y(t) \) to
\[ y'' + 4y = \frac{3}{\sin(2t)}, \quad 0 < t < \frac{\pi}{2}. \]
We make the ansatz
\[ y(t) = u_1(t) \cdot \sin(2t) + u_2(t) \cdot \cos(2t) \]
for unknown coefficients \( u_1(t) \), \( u_2(t) \)
satisfying
\[ u_1'(t) \cdot \sin(2t) + u_2'(t) \cdot \cos(2t) = 0. \]

The Wronskian of \( \frac{1}{2} \sin(2t) \), \( \cos(2t) \) is
\[ W(\sin(2t), \cos(2t)) = \begin{vmatrix} \sin(2t) & \cos(2t) \\ 2 \cos(2t) & -2 \sin(2t) \end{vmatrix} \]
\[ = -2 \sin^2(2t) - 2 \cos^2(2t) \]
\[ = -2 \cdot (\sin^2(2t) + \cos^2(2t)) \]
\[ = -2. \]

Thus, we obtain for \( u_1(t) \) and \( u_2(t) \):
\[ u_1(t) = -\int \frac{\cos(2t) \cdot \frac{3}{2}}{-2} dt \]
\[ = +\frac{3}{2} \int \frac{\cos(2t)}{\sin(2t)} dt \]
\[ = \frac{3}{4} \ln(\sin(2t)) \]

and
\[ u_2(t) = +\int \frac{\sin(2t) \cdot \frac{3}{2}}{-2} dt \]
\[ = -\frac{3}{2} \int 1 dt \]
\[ = -\frac{3}{2} t \]

Note that we can drop the unknown constants here!
Hence, the general solution is given by
\[ y(t) = c_1 \cdot \sin(2t) + c_2 \cdot \cos(2t) \
+ \frac{3}{4} \ln(\sin(2t)) \cdot \sin(2t) - \frac{3}{2} t \cdot \cos(2t) \]

In practice, the method of variation of parameters can have two caveats:

1. It can be difficult to determine explicitly a fundamental set of solutions \( y_1(t), y_2(t) \) to the associated homogeneous equation \( Ly = 0 \) (when the coefficients are non-constant).

2. It can be difficult to evaluate the integrals for \( u_1(t) \) and \( u_2(t) \) explicitly.

Read Section 3.7 on Mechanical and Electrical Vibrations to learn about some important physical processes that are modeled by 2nd order linear ODEs with constant coefficients.