1. \((10 \text{ points})\) Determine the general solution to the ODE

\[
(3 - \ln(x)) \frac{dy}{dx} = \frac{y}{x} + 4x, \quad x > 0.
\]

**Solution.** We rearrange the equation as

\[
-\frac{y}{x} - 4x + (3 - \ln(x)) \frac{dy}{dx} = 0.
\]

Writing \(M(x, y) = -\frac{y}{x} - 4x\) and \(N(x, y) = 3 - \ln(x)\), we compute that

\[
\frac{\partial M}{\partial y}(x, y) = -1 = \frac{\partial N}{\partial x}(x, y).
\]

Hence, this first order ODE is exact and there must exist a function \(\psi(x, y)\) with the property that

\[
\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \quad \frac{\partial \psi}{\partial y}(x, y) = N(x, y).
\]

In order to determine \(\psi(x, y)\), we first integrate the identity \(\frac{\partial \psi}{\partial x}(x, y) = M(x, y)\) with respect to \(x\) to find that

\[
\psi(x, y) = \int M(x, y) \, dx = \int -\frac{y}{x} - 4x \, dx = -y \ln(x) - 2x^2 + h(y)
\]

for some unknown function \(h(y)\) that only depends on \(y\). To determine \(h(y)\) we exploit that we must have that \(\frac{\partial \psi}{\partial y} = N(x, y)\) and therefore

\[
-\ln(x) + h'(y) = \frac{\partial \psi}{\partial y} = N(x, y) = 3 - \ln(x),
\]

which implies that

\[
h'(y) = 3.
\]

Thus, we may take \(h(y) = 3y\). It follows that the general solution to the ODE is given by

\[
y\ln(x) - 2x^2 + 3y = C
\]

or

\[
y(3 - \ln(x)) = 2x^2 + C
\]

for some unknown constant \(C\).

(Alternatively, one can also determine the general solution using the method of integrating factors, since this is a first-order linear ODE)
2. (10 points) Solve the initial value problem

\[(1 + t^2) \frac{dy}{dt} = t \cos(t^2) - 2ty, \quad y(0) = 2.\]

**Solution.** This is a first-order linear equation and we will solve it by the method of integrating factors. We divide both sides by $1 + t^2$ and write the result as:

\[
\frac{dy}{dt} + \frac{2t}{1 + t^2} y - \frac{t}{1 + t^2} \cos(t^2) = 0.
\]  

(1)

An integrating factor for this is:

\[\exp \left( \int \frac{2t}{1 + t^2} dt \right) = \exp \log(1 + t^2) = 1 + t^2.\]

Multiplying both sides of (1) by $1 + t^2$ gives:

\[\frac{d}{dt} \left( (1 + t^2)y \right) = t \cos(t^2).\]

(In fact, one can see this directly from the given equation without first passing to (1)). Integrating both sides gives:

\[(1 + t^2)y = \int t \cos(t^2) dt = \frac{1}{2} \sin(t^2) + C,\]

where we made the substitution $u = t^2$ to evaluate the integral and where $C$ is a constant of integration. Therefore, the general solution is:

\[y(t) = \frac{1}{2(1 + t^2)} \sin(t^2) + \frac{C}{1 + t^2}.\]

To satisfy the condition $y(0) = 2$, we must take $C = 2$, so the solution to the initial value problem is:

\[y(t) = \frac{1}{2(1 + t^2)} \sin(t^2) + \frac{2}{1 + t^2}.\]
3. (10 points) Solve the initial value problem
\[ y'' + y' - 6y = 0, \quad y(0) = a, \quad y'(0) = 1, \]
where \( a \in \mathbb{R} \) is a real number. Find all values of \( a \) for which the solution \( y(t) \) tends to \(+\infty\) as \( t \to \infty \).

Solution. This is a second order linear homogeneous ODE. The associated characteristic equation
\[ r^2 + r - 6 = 0, \]
\[ (r + 3)(r - 2) = 0, \]
has two distinct real roots \( r_1 = -3 \) and \( r_2 = 2 \). Thus, the general solution to this ODE is given by
\[ y(t) = c_1 e^{-3t} + c_2 e^{2t} \]
for arbitrary constants \( c_1, c_2 \in \mathbb{R} \). To satisfy the initial conditions \( y(0) = a, y'(0) = 1 \), we must have that
\[ c_1 + c_2 = a \]
\[ -3c_1 + 2c_2 = 1. \]
This implies that \( c_1 = \frac{2a - 1}{5} \) and \( c_2 = \frac{3a + 1}{5} \). The solution to the initial value problem is therefore given by
\[ y(t) = \frac{2a - 1}{5} e^{-3t} + \frac{3a + 1}{5} e^{2t}. \]
Since \( e^{-3t} \to 0 \) as \( t \to \infty \) and \( e^{2t} \to +\infty \) as \( t \to \infty \), in order to have that \( y(t) \to +\infty \) as \( t \to \infty \), the coefficient \( \frac{3a + 1}{5} \) in front of \( e^{2t} \) must be strictly positive
\[ \frac{3a + 1}{5} > 0. \]
Hence, we must have \( a > -\frac{1}{3} \).
4. (10 points) Solve the initial value problem
\[ y \frac{dy}{dt} - at = aty^2, \quad y(0) = 1, \]
where \( a \in \mathbb{R} \) is a real number. Determine the interval in which the solution is defined.

Solution. This is a separable first order ODE. Adding \( at \) to both sides, dividing by \((1 + y^2)\) and integrating, it becomes
\[ \int \frac{y}{1 + y^2} dy = \int at \, dt. \]
This gives
\[ \frac{1}{2} \log(1 + y^2) = \int at \, dt = \frac{1}{2} at^2 + C_1, \]
where \( C_1 \) is a constant of integration. Rearranging, we have that \( y = y(t) \) satisfies
\[ y^2 = Ce^{at^2} - 1 \]
where \( C = e^{2C_1} \). To satisfy \( y(0) = 1 \) we must take \( C = 2 \), thus we find that
\[ y(t) = (2e^{at^2} - 1)^{1/2}. \]
(Note that we have taken the positive square root since \( y(0) = 1 > 0 \).) For this to be valid, we need \( 2e^{at^2} - 1 \geq 0 \). If \( a \geq 0 \), this holds for all \( t \), so in this case the maximal interval of existence is \(( -\infty, \infty )\). If instead \( a < 0 \), we need \( at^2 \geq \ln(1/2) \), or \( t^2 \leq \frac{\ln(1/2)}{a} = \frac{\ln(2)}{-a} \) and so the maximal interval of existence is \( \left[ -\sqrt{\frac{\ln(2)}{-a}}, +\sqrt{\frac{\ln(2)}{-a}} \right] \).
5. (a) (4 points) The functions \( y_1(t) = e^t \cos(2t) \) and \( y_2(t) = e^t \sin(2t) \) are solutions to the ODE
\[
y'' - 2y' + 5y = 0.
\]
Is \( \{y_1, y_2\} \) a fundamental set of solutions for this ODE?

Solution.
By a theorem from class, \( y_1, y_2 \) form a fundamental set of solutions for this ODE if their Wronskian \( W(y_1, y_2)(t) \) is nonzero for all \( t \). We compute:
\[
y'_1 = e^t \cos(2t) - 2e^t \sin(2t), \quad y'_2 = e^t \sin(2t) + 2e^t \cos(2t),
\]
so:
\[
W(y_1, y_2)(t) = y_1(t)y'_2(t) - y'_1(t)y_2(t)
= e^t \cos(2t)(e^t \sin(2t) + 2e^t \cos(2t)) - e^t \sin(2t)(e^t \cos(2t) - 2e^t \sin(2t))
= e^{2t} \left( \cos(2t) \sin(2t) + 2 \cos^2(2t) - \sin(2t) \cos(2t) + 2 \sin^2(2t) \right)
= 2e^{2t},
\]
using \( \sin^2(2t) + \cos^2(2t) = 1 \). This is nonzero for all \( t \), so \( y_1, y_2 \) are a fundamental set of solutions.
(b) (6 points) Let \( f(t) = 10 + \sin(t) \) and let \( g(t) \) be an unknown function satisfying \( g(0) = 10 \). If the Wronskian of \( f(t) \) and \( g(t) \) is

\[
W(f, g)(t) = t^2 (10 + \sin(t))^2
\]

find \( g(t) \).

**Solution.** The Wronskian of \( f, g \) is:

\[
W(f, g)(t) = f(t) g'(t) - f'(t) g(t),
\]

so the function \( g \) satisfies the equation:

\[
(10 + \sin(t)) g'(t) - \cos(t) g(t) = t^2 (10 + \sin(t))^2.
\]

This is a first-order linear ODE and we can solve it using the method of integrating factors. We divide both sides by \( 10 + \sin(t) \) and it becomes:

\[
g'(t) - \frac{\cos(t)}{10 + \sin(t)} g(t) = t^2 (10 + \sin(t)).
\]

An integrating factor for this equation is:

\[
\exp\left( \int -\frac{\cos(t)}{10 + \sin(t)} \, dt \right) = \exp(- \log(10 + \sin(t))) = (10 + \sin(t))^{-1},
\]

and multiplying both sides of the equation by this leads to:

\[
\frac{d}{dt} ((10 + \sin(t))^{-1} g(t)) = t^2.
\]

Integrating both sides, we get:

\[
(10 + \sin(t))^{-1} g(t) = \frac{1}{3} t^3 + C,
\]

for an unknown constant \( C \). Since \( g(0) = 10 \), from this expression we get \( C = 1 \), so the function \( g(t) \) is:

\[
g(t) = (10 + \sin(t)) \left( \frac{1}{3} t^3 + 1 \right).
\]
6. Consider the autonomous first-order ODE

\[
\frac{dy}{dt} = y^2 - 2y + a,
\]

where \( a \in \mathbb{R} \) is a parameter.

(a) (4 points) Draw the phase line for the parameter value \( a = 0 \). Determine the critical points and their stability.

Solution. For the parameter value \( a = 0 \) the right-hand side of this ODE is given by \( y^2 - 2y = y(y - 2) \), which has roots at \( y = 0 \) and at \( y = 2 \). Please see the end of this file for a sketch of the phase line. It follows that the critical point \( y = 0 \) is stable, while the critical point \( y = 2 \) is unstable.
(b) (6 points) Sketch a bifurcation diagram for this ODE in the \( a-y \)-plane. Indicate the stability of the critical points. What are the bifurcation values of \( a \)?

Solution. The critical points are the real roots of the equation

\[
y^2 - 2y + a = 0.
\]

By completing the square or by using the quadratic formula, it follows that the roots are given by

\[
y = 1 \pm \sqrt{1 - a}.
\]

It follows that the equation \( y^2 - 2y + a = 0 \) has no real roots for \( a > 1 \), it has exactly one real root \( y = 1 \) for \( a = 1 \), and for \( a < 1 \) it has two distinct real roots given by \( y_1 = 1 - \sqrt{1 - a} \) and \( y_2 = 1 + \sqrt{1 - a} \).

Please see the end of this file for a sketch of the bifurcation diagram for this ODE. It follows that \( a = 1 \) is a bifurcation value, because the number of critical points of the ODE changes from two to zero as the parameter value \( a \) passes through \( a = 1 \).
Problem 6

(a)

\[ y(\frac{dy}{dx}) - y^2 = y^2 - 2y \]

phase line:

\[ 0 \quad 2 \]
\( y = 1 + \sqrt{1 - \alpha} \)

\( y = 1 - \sqrt{1 - \alpha} \)

\( \alpha > 1: \)

\( y^2 - 2y + \alpha \)

\( \alpha = 1: \)

\( y^2 - 2y + 1 \)

\( \alpha < 1: \)

\( 1 - \sqrt{1 - \alpha} \)

\( 1 + \sqrt{1 - \alpha} \)