A system of \( n \) linear (algebraic) equations

\[
a_{11} \cdot x_1 + \ldots + a_{1n} \cdot x_n = b_1 \\
\vdots \\
a_{n1} \cdot x_1 + \ldots + a_{nn} \cdot x_n = b_n
\]

can be written as a matrix equation

\[
A \cdot \vec{x} = \vec{b},
\]

where \( A \) is the \( n \times n \) matrix

\[
A = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{bmatrix},
\]

\( \vec{x}, \vec{b} \) denote the column vectors

\[
\vec{x} = \begin{bmatrix} x_1 \\
\vdots \\
x_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\
\vdots \\
b_n \end{bmatrix},
\]

and \( A \cdot \vec{x} \) denotes matrix multiplication.

\( \Rightarrow \) All vectors are, by convention, considered column vectors.
Properties of matrices and matrix equations

1. If \( E = \begin{bmatrix} 0 \\ \vdots \end{bmatrix} = \bar{0} \) in \( A \bar{x} = \bar{b} \),
   the equation is homogeneous, otherwise non-homogeneous.

2. The determinant \( \det(A) \) of the \( nxn \) matrix \( A \) is a useful value that can be computed from the entries of \( A \).
   For a \( 2 \times 2 \) matrix:
   \[
   \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
   \]
   For a \( 3 \times 3 \) matrix:
   \[
   \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot e \cdot i + b \cdot f \cdot g + c \cdot d \cdot h
   \]
   \[-c \cdot e \cdot g - a \cdot f \cdot h - b \cdot d \cdot i\]
   \[
   + c \cdot e \cdot g - a \cdot f \cdot h - b \cdot d \cdot i
   \]
   \[
   = a \cdot \begin{vmatrix} e & f \\ g & h \end{vmatrix} - d \cdot \begin{vmatrix} c & f \\ h & i \end{vmatrix} + g \cdot \begin{vmatrix} c & e \\ h & i \end{vmatrix}
   \]

3. If \( \det(A) \neq 0 \), then the system \( A \bar{x} = \bar{b} \) has a unique solution.
   In particular, then \( A \bar{x} = \bar{0} \) has the unique solution \( \bar{x} = \bar{0} = \begin{bmatrix} 0 \\ \vdots \end{bmatrix} \).
   If \( \det(A) = 0 \), then \( A \bar{x} = \bar{0} \) has many solutions...
If \( \det(A) \neq 0 \), then the inverse matrix of \( A \), \( A^{-1} \), exists with the property:

\[
A^{-1} \cdot A = I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

then we can "solve" \( A\hat{x} = \hat{b} \):

\[
A\hat{x} = \hat{b} \Rightarrow A^{-1} \cdot A\hat{x} = A^{-1} \hat{b} \\
\hat{x} = A^{-1} \hat{b}
\]

In practice, to solve \( A\hat{x} = \hat{b} \), we use the elementary row operations to reduce the system into a much simpler form, from which the solutions can be read off (if there are any):

(i) interchanging two rows
(ii) multiplication of a row by a non-zero scalar
(iii) addition of any multiple of one row to another row
To do this, form the augmented matrix
\[
A \mid b = \begin{bmatrix}
  a_{11} & \cdots & a_{1n} & | & b_1 \\
  \vdots & \ddots & \vdots & | & \vdots \\
  a_{m1} & \cdots & a_{mn} & | & b_m
\end{bmatrix}.
\]

**Definition:**
A set of \( k \) vectors \( \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)} \) is **linearly dependent** if there exist real numbers \( c_1, \ldots, c_k \in \mathbb{R} \), not all zero, such that
\[
c_1 \cdot \mathbf{x}^{(1)} + \cdots + c_k \cdot \mathbf{x}^{(k)} = \mathbf{0}.
\]
Otherwise, they are **linearly independent**.

**Observe:** Given an \( n \times n \) matrix \( A \), then the \( n \) column vectors of \( A \) are linearly independent if and only if \( \det(A) \neq 0 \).
We denote by $\mathbb{R}^n$ the set of all $n$-vectors. Then we can view the equation $A\overline{\mathbf{x}} = \overline{\mathbf{b}}$ as a linear transformation of $\mathbb{R}^n$ that maps $\overline{\mathbf{x}} \in \mathbb{R}^n$ to $\overline{\mathbf{b}} \in \mathbb{R}^n$:

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\overline{\mathbf{x}} \rightarrow \overline{\mathbf{b}} = A\overline{\mathbf{x}}.$$

### Eigenvalues and eigenvectors

Special vectors that are transformed into multiples of themselves are called eigenvectors. They must satisfy

$$A\overline{\mathbf{x}} = \lambda \overline{\mathbf{x}}$$

for some scalar $\lambda$, called its eigenvalue. We have

$$A\overline{\mathbf{x}} = \lambda \overline{\mathbf{x}} \iff (A - \lambda \cdot \mathbb{I}_n)\overline{\mathbf{x}} = \mathbf{0}.$$  

This equation only has non-zero solutions if

$$\det (A - \lambda \cdot \mathbb{I}_n) = 0.$$ 

This equation is called the characteristic equation of the matrix $A$, it is a polynomial equation of degree $n$ in $\lambda$. 

\[\text{---125---}\]
Example: Find the eigenvalues and eigenvectors of the matrix

\[ A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}. \]

Solution:

- Characteristic equation:

\[ 0 = \text{det}(A - \lambda I) \]

\[ = \begin{vmatrix} 5 - \lambda & -1 \\ 3 & 1 - \lambda \end{vmatrix} \]

\[ = (5 - \lambda)(1 - \lambda) - 3(-1) \]

\[ = 5 - 6\lambda + \lambda^2 + 3 \]

\[ = \lambda^2 - 6\lambda + 8 \]

\[ = (\lambda - 2)(\lambda - 4) \]

\[ \Rightarrow \lambda = 2 \text{ and } \lambda = 4 \text{ are the eigenvalues of } A \]

- Eigenvectors for \( \lambda = 2 \):

\[(A - 2I)x = 0 \]

\[ \Rightarrow \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \]

\[ \Rightarrow 3x_1 - x_2 = 0 \Rightarrow x_2 = 3x_1 \]

Let \( x_1 = c \)

\[ \Rightarrow x = c \begin{bmatrix} 1 \\ 3 \end{bmatrix} \] are the eigenvectors for the eigenvalue \( \lambda = 2 \).
Eigenvectors for $\lambda = 4$:

$$(A - 4I) \mathbf{v} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow x_1 - x_2 = 0$$

Let $x_1 = c$

$$\Rightarrow \mathbf{v}(2) = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are the eigenvectors for $\lambda = 4$.

These are the only two directions that are left invariant by A!