Solutions for Midterm 1

1 a. By D’Alembert’s formula:
\[ u(x, t) = 0 + \frac{1}{2c} \int_{x-ct}^{x+ct} s e^{-s^2} \, ds = -\frac{1}{4c} e^{-s^2} \bigg|_{x-ct}^{x+ct} = \frac{1}{4c} [e^{-(x-ct)^2} - e^{-(x+ct)^2}] \]

1 b. Using the formula above, \( u(x, \frac{100}{t}) = \frac{1}{4c}[e^{-(x-100)^2} - e^{-(x+100)^2}] \), a reasonable sketch will include a “bump” above the x-axis at \( x = 100 \) and a bump below the x-axis at \( x = -100 \). The best sketches will show that the bumps have width 1 and their peaks just barely hit \( \pm \frac{1}{4c} \).

2. Computing the derivative of \( \tilde{E}(t) \) yields
\[
\frac{d\tilde{E}(t)}{dt} = \frac{1}{2} \int_0^L \frac{\partial}{\partial t} [u(x, t)]^2 \, dx = \int_0^L u(x, t) \frac{\partial u}{\partial t}(x, t) \, dx = k \int_0^L u(x, t) \frac{\partial^2 u}{\partial x^2}(x, t) \, dx - \alpha \int_0^L u(x, t) u(x, t) \, dx
\]
\[= k \int_0^L \left[ \frac{\partial u}{\partial x}(x, t) \right]^2 \, dx - \alpha \int_0^L [u(x, t)]^2 \, dx \leq 0 \]

hence \( \tilde{E}(t) \) is a decreasing function of \( t \) and that \( \tilde{E}(t) \leq \tilde{E}(0) = \frac{1}{2} \int_0^L [u(x, 0)]^2 \, dx = \frac{1}{2} \int_0^L [f(x)]^2 \, dx \).

A second approach would be to set \( u(x, t) = e^{-\alpha t} v(x, t) \) and apply the energy method to \( v(x, t) \), which now satisfies a heat equation without sources. This approach has the advantage of showing that energy actually decays exponentially \( \tilde{E}(t) \leq e^{-2\alpha t} \tilde{E}(0) \).

3. This is due to the maximum (and minimum) principle. Consider the rectangle
\[ R = \{(x, t) : 0 \leq x \leq 2, 0 \leq t \leq 2 \} \]

Let \( \Gamma \) denote the union of the 2 lateral sides with the initial side of the boundary of \( R \) \( \{ \{x = 0\}, \{x = 2\}, \{t = 0\} \} \). The maximum and minimum principles tell us that
\[ \max_{(x,t) \in R} u(x, t) = \max_{(x,t) \in \Gamma} u(x, t), \quad \min_{(x,t) \in R} u(x, t) = \min_{(x,t) \in \Gamma} u(x, t). \]

On the initial side, \( 1 - (x - 1)^2 \) is a downward parabola with zeros at \( x = 0, 2 \) and max at \( x = 1 \) which tells us \( 0 \leq 1 - (x - 1)^2 \leq 1 \). On the lateral sides, \( u(0, t) = 0 \) and \( u(2, t) = t \), so \( 0 \leq u(2, t) \leq 2 \) when \( 0 \leq t \leq 2 \) and \( u(2, 2) = 2 \). Hence the max on the boundary is 2 and the min is 0. Applying the max/min principles tells us that \( 0 \leq u(x, t) \leq 2 \) for any \( (x, t) \) in \( R \) (actually the strong version of the principle tells us we can replace this with strict inequalities on the interior of \( R \)).

4. Solution 1: First form the full Fourier series corresponding to initial data \( u(x, 0) = \cosh(x) \)
\[ \cosh(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx) \]
where \( A_0, A_n, B_n \) are given by the standard formulas (actually, \( \cosh(x) \) is continuous and \( \cosh(\pi) = \cosh(-\pi) \), so by theorem we can replace the ‘\( \sim \)’ by an equality ‘\( = \)’). Hence the solution to the problem is given by

\[
u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)e^{-n^2t} + \sum_{n=1}^{\infty} B_n \sin(nx)e^{-n^2t}.\]

We should then have

\[
\lim_{t \to \infty} u(x, t) = A_0 + \lim_{t \to \infty} \left[ \sum_{n=1}^{\infty} A_n \cos(nx)e^{-n^2t} + \sum_{n=1}^{\infty} B_n \sin(nx)e^{-n^2t} \right] = A_0 + 0 = A_0,
\]

since each term in both series decays exponentially to 0 as \( t \to \infty \). The equilibrium solution is thus constant and equal to

\[
A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh(x)dx = \frac{1}{2\pi} \sinh(x)\left|_{-\pi}^{\pi} \right. = \frac{1}{2\pi} (\sinh(\pi) - \sinh(-\pi)) = \frac{1}{\pi} \sinh(\pi).
\]

Solution 2: Consider the equilibrium problem \( v''(x) = 0, v(\pi) = v(-\pi), v'(\pi) = v'(-\pi) \). It is easy to see that \( v(x) \) must be a line, and by considering boundary conditions, it must be a horizontal line, i.e. \( v(x) = c \) for some constant \( c \). Now consider the energy functional \( E(t) = \int_{-\pi}^{\pi} u(x, t)dx \). This sort of functional is actually conserved for all \( t \geq 0 \) as

\[
\frac{dE(t)}{dt} = \int_{-\pi}^{\pi} \frac{\partial}{\partial t} (x, t)dx = \int_{-\pi}^{\pi} \frac{\partial^2 u}{\partial x^2}(x, t)dx = \frac{\partial u}{\partial x}(x, t)|_{-\pi}^{\pi} = 0
\]

by the boundary conditions. Hence

\[
\int_{-\pi}^{\pi} \cosh(x)dx = E(0) = \lim_{t \to \infty} E(t) = \int_{-\pi}^{\pi} v(x)dx = \int_{-\pi}^{\pi} Cdx = C2\pi
\]

which implies \( v(x) = C = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh(x)dx = \frac{1}{\frac{1}{\pi}} \sinh(\pi) \).

5. Separating variables \( u(x, t) = G(t)\phi(x) \) gives

\[
\frac{G''(t)}{G(t)} - 4 = \frac{\phi''(x)}{\phi(x)} = -\lambda, \quad \phi''(x) = -\lambda \phi(x), \quad G'(t) = (4 - \lambda)G(t)
\]

As noted in the hint, \( \phi(x) = \sin(nx), \lambda = n^2 \), meaning \( G(t) = C \exp(4t - n^2t) \). Hence our family of separated solutions look like \( u(x, t) = C_n e^{4t-n^2t} \sin(nx) \) and correspond to initial data of the form \( C_n \sin(nx) \).

Part a): \( u(x, t) \) is just a separated solution with \( C_1 = 5 \), in other words

\[
u(x, t) = 5e^{4t-n^2t} \sin(x) = 5e^{3t} \sin(x)
\]

Hence \( \lim_{t \to \infty} u(x, t) = \infty \) provided \( \sin(x) > 0 \), but this is true for any \( 0 < x < \pi \).

Part b): \( u(x, t) \) is now just a sum of 2 separated solutions

\[
u(x, t) = 3e^{4t-5^2t} \sin(5x) - 10e^{4t-2^2t} \sin(2x) = 3e^{-21t} \sin(5x) - 10 \sin(2x).
\]

Hence \( \lim_{t \to \infty} u(x, t) = 0 - 10 \sin(2x) = -10 \sin(2x) \).