Math 421: Dynamical Systems & Chaos – Professor Haskins

Homework 1: 13 Sep 00
Due: 18 Sep 00 at start of class

If you haven’t done this already please do the following
send me an email (mhaskin@math.jhu.edu) containing (in this order):
• Your name and the email address you want to use for this class
• A “name” to identify yourself on any class roster which shouldn’t reveal
  your real name
• Your year, major and any minors
• Any math classes you have taken at Hopkins and dates
• Some reasons why you decided to take this class
• Anything particular you hope to learn in this class
• Any mathematical software you are familiar with

Remember to show all your working and to write clearly, neatly.

Read chapters 2 and 3 of the photocopied notes from Devaney’s book “A
First Course in Chaotic Dynamical Systems”.

1. (a) For each of the following functions \( f \) find \( f^2(x) \) and \( f^3(x) \) (\( f \) composed with itself twice and three times, not \( f \) differentiated)
   (i) \( f(x) = 2x \)
   (ii) \( f(x) = 3x - 2 \)
   (iii) \( f(x) = x^2 - 3 \)
   (iv) \( f(x) = \sqrt{x + 1} \)
   (v) \( f(x) = 2^x \).
   (b) For each of the functions in part (a) compute \( f^5(0) \). If you have difficulty, explain why.
   (c) (Harder) Try to find expressions for \( f^n(x) \) for each of the functions in
   part (a). This will not be possible for some of the functions. Try to simplify
   your answer as much as possible.

2. Consider the discrete time dynamical system

\[
x(t + 1) = ax(t), \quad x(0) = b
\]

where \( a \) and \( b \) are real numbers.
   (i) Find a formula for \( x(t) \) in terms of \( a \) and \( b \).
   (ii) Describe qualitatively the different possible types of dynamical behaviour
this system can exhibit for different values of $a$ and $b$. Draw graphs to illustrate.

3. Consider the continuous dynamical system

\[ x' = 3x, \quad x(0) = 2. \]

(a) Verify that $x(t) = 2e^{3t}$ solves this system.
(This means you need to do two things. One: check that this function satisfies the differential equation. Two: check that it satisfies the initial condition.)

(b) Draw a graph of this solution plotting $x$ on the $y$-axis and $t$ on the $x$-axis.

4. (i) Using the previous question to help you, find a formula for $x(t)$ for the system

\[ x' = ax, \quad x(0) = b, \]

where $a$ and $b$ are fixed real numbers.
(ii) Check that your answer is correct.
(iii) What happens in the special cases $a = 0$ or $b = 0$?
(iv) Describe qualitatively the different possible types of dynamical behaviour this system can exhibit for different values of $a$ and $b$. Draw graphs to illustrate.

5. Compare the behaviour of the discrete dynamical system you studied in question 2 with the analogous continuous one in question 4. What do you notice about their qualitative dynamics?

6. You have $1000 in your bank account on Aug 1. At the beginning of each subsequent month you put in an extra $100. Your bank pays a monthly interest rate of $r\%$.
(i) Write down a discrete time dynamical system that relates how much money you have in your account at one month to how much you have the next month.

(ii) Use the formula from part (i) to find a formula for how much money you will have $k$ months from now.

(iii) Calculate how much money will be in your account in 10 and 20 years respectively if your bank pays 0.5% interest per month (you will need a calculator to evaluate this).

(iv) Repeat part (iii) assuming that your bank pays 0.75% per month.

(v) Compare you answers from parts (iii) and (iv).