Discs and Tangles

Tangles were first explored in the 1960s by John Conway as an algebraic notion. They have been useful tools for understanding knots. Perhaps their most famous application has been to aid in rigorously understanding DNA-protein interactions.

The (so far, unsolved!) problem here is to find a concise way of keeping track of the 'complexity' of the tangle, in terms of a special disc. The solution will require a mixture of calculations, visualization and proof.

**Background:**

**Def:** A tangle (written \((B, t)\)) is a three-dimensional ball \(B\), containing two disjoint arcs that are represented by \(t\), such that the endpoints of the arcs lie on the boundary of \(B\). These 4 endpoints are typically labelled NW, NE, SW and SE.

**Ex:** Here are some diagrams of tangles, from left to right, the \((-3)\) tangle, and two diagrams of the trivial infinity tangle \(\infty\):

---

**Def:** Two tangles are said to be equivalent if you can move the arcs in one into the pattern of the other *without* moving the endpoints.

So above, the second and third tangles are equivalent.

There is a special class of tangles that will be our focus, called rational tangles.

**Def:** A tangle \((B, t)\) is said to be rational if you can move (without any self-intersection) the arcs from the middle tangle above to the arcs in your tangle. You are allowed the endpoints to move on the boundary of \(B\), but you cannot pull them off the boundary.

**Ex:** The first and second are rational tangles. The third and fourth are not.
In fact, Conway showed exactly how to go from the rational tangle of your choice to the infinity tangle. Here’s how, for the first rational tangle above:

1. In order to untwist the right side of the tangle, we move the two endpoints SE and NE around in a full clockwise circle in the Eastern hemisphere, so that we have this diagram:

2. Then we move the two endpoints, which are now in positions SW and SE, around in a full counterclockwise circle in the Southern hemisphere to obtain our ∞ tangle.

In general, we will always undo the rational tangle by starting with the right side of the tangle and untwisting the endpoints labeled SE and NE some number of times (possibly zero), say $a_n$. (We call this undoing the horizontal twists.) Then we will untwist the endpoints labeled SE and SW some number of times, say $a_{n-1}$ (undoing the vertical twists), after which we further untwist ($a_{n-2}$ times) the endpoints labeled SE and NE, and so on, as many times as is necessary until we get the infinity tangle or the infinity tangle rotated by 180 degrees.

Note that that we can be untwisting a positive (clockwise) or negative (negative) number of times.

In this language, then, we can rephrase our example. We undo 2 horizontal twists, followed by −2 vertical twists.

If we write our rational tangle as a vector $(a_1, a_2, ..., a_n)$, our example then becomes $(-2, 2)$.

The reason rational tangles are called rational, is that from every vector representing a rational tangle we can associate a unique rational number $b/a$ (we allow ∞ = 1/0 to be a rational number!), by using a continued fraction. We define $b/a$ by the property that

$$b/a = a_n + 1/(a_{n-1} + 1/(... + 1/a_1))$$

So our tangle example becomes $b/a = 2 + 1/(-2) = 3/2$.

Another alternative definition of a rational tangle is that there exists a disc, $D$, with boundary on the boundary of our tangle, where $D$ separates the two arcs of $(B, t)$.

**Ex:** For the infinity tangle, the disc $D$ is just a vertical disc that bisects the ball.

**The Question**
Say you start with the infinity tangle and the disc $D$, and you start making your favorite rational tangle, $b/a$, by the reverse of the process above. (That is, you write out the continued fraction for $b/a = (a_1, a_2, ..., a_n)$. Then, with the diagrams you introduce a certain number of vertical twists, followed by a given number of horizontal twists, etc.) (Question: how many vertical or horizontal?)

What happens to the disc?
That is to say, can you write a concise expression (in terms of the number of horizontal and vertical twists) for how many times the disc twists around the boundary of $B$ both horizontally and vertically?

Alternative Equivalent Question

Background
The equivalence between the question above and the question below is based on the notion of double branched covers. These are not difficult ideas, but do require a fair bit of technical jargon. If you are interested in this problem a more information can be found in Erica Flapan's book (see below).

We now turn to the equivalent question, stated in terms of an annulus on a torus.

Def: The 2-dimensional surface of a donut is called a torus. More formally, it can be described by $S^1 \times S^1$.

Def: The 2-dimensional surface of a punctured disc is called an annulus. More formally, it can be described by $S^1 \times I$.

It is a convenient fact that our torus can be decomposed into two annuli, with a shared common boundary of two parallel curves. These curves closes back on itself without any self-intersections.

Ex:
For any such (closed and without self-intersections) curve $r$, we can refer to this in terms of a basis $\mu$ and $\lambda$, i.e., we can write $r = c\mu + d\lambda$ such that $c$ and $d$ are integers.

**Def:** $\mu$ is the **meridional curve**, a circle with no self intersections on our torus that bounds a disc in three-dimensional space. $\lambda$ is a **longitudinal curve**, a circle on our torus (with no self-intersections) that intersects $\mu$ exactly once. There are many such curves, but for convenience we pick the one shown below.

**Ex:** For our parallel boundary curves in the example above, they wrap three times meridionally and two times longitudinally, so they can be written as $(3, 2)$ curves.

There are two interesting ways to play with this torus.
1. Cut the torus along a meridional curve, twist one side meridionally by $n$ $1/2$ twists, and then reglue.
   **Ex:** Twisting 2 times meridionally.
2. Cut torus along a longitudinal curve and its shadow on the underside (this is often likened to cutting the donut with a biscuit cutter), twist one piece longitudinally by \( n \) 1/2 twists and reseal.
   \textbf{Ex:} Twisting 2 times longitudinally.

\textbf{The Question for an Annulus}

From the theory of double branched covers, tracking the boundary of the disc for the tangle corresponds to tracking the boundary curve \( r \) of a certain annulus for the torus.

Pictorially, given the following tangle diagram, with arcs and disc as as below, we yield the corresponding torus with arcs and annulus as shown:

So horizontal twists of the tangle arcs correspond to meridional twists of the torus, and vertical twists of the tangle arcs correspond to longitudinal twists. We can thus rephrase the process of going from the \( \infty \) tangle to a given rational tangle in terms of a finite alternating sequence of meridional and longitudinal twists of the torus.

\textbf{Can you describe what happens to the annulus curve} \( r \) \textbf{as you create the rational tangle} \( b/a \)?

\textbf{Warmup Questions}

1. Draw the following rational tangles: \((1), (-1), (1,0), (-1,0), (3,2,1), (3,2,1,0), (1,2,3,0)\).
2. Let \( b = 2 \) and \( a = 3 \). Write the tangle vector associated with \( b/a \) and draw the tangle that has that vector.
3. Draw $(2, 3), (1, 1), (1, 0), (0, 1), (5, 2)$ curves on torii.

3. As always, you might want to start with a few simple examples:

**Background reading:**


(If you are interested in this problem, please let me know and I will photocopy the relevant pages.)