The Schrödinger Equation

with a

Non-Smooth Magnetic Potential

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[Some joint work with M. Burak Erdogan, Wilhelm Schlag]

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Free Schrödinger Equation \((in \, \mathbb{R}^n)\)

\[
\begin{aligned}
\begin{cases}
  i \partial_t u(t,x) = -\Delta u(t,x) \\
  u(0,x) = u_0(x)
\end{cases}
\end{aligned}
\]

\[u(t,x) = e^{it\Delta} u_0. \quad \text{Unitary } \Rightarrow \mathcal{L}^2 \text{ conservation.}\]

Fourier Inversion: \[U(t,x) = \frac{1}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{i|x-y|^2}{4t}} u_0(y) \, dy\]

Dispersive estimate \[\|u(t,\cdot)\|_p \leq |t|^{-\frac{n}{2} \left(\frac{1}{p} - \frac{1}{q}\right)} \|u_0\|_p.\]

TT* argument + Hardy-Littlewood-Sobolev

\[\|u(t,x)\|_{L_t^p L_x^q} \leq \|u_0\|_{L_x^2} \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad 2 \leq p, q \leq \infty, \quad \text{unless } n=2.\]
Magnetic Schrödinger Equation (time-independent)

\[ i \partial_t u(t,x) = \left( -\Delta + \frac{i}{2} (A_0 \cdot \nabla + \nabla \cdot A_0) + V_0 \right) u(t,x) \]
\[ = (-\Delta + L(x)) u(t,x) \]
\[ = H u(t,x) \]

**Question:** Is it true that \( \| e^{-itH} u_0 \|_{L_t^p L_x^q} \leq \| u_0 \|_2 \)?

**Answer:** Not always. If \( H \) has eigenvalue \( \lambda \), then \( u(t,x) = e^{-it \lambda} \psi(x) \) is solution. 

Strichartz inequality is only satisfied for \( p = \infty, q = 2 \).

**Better Question:** For what class of \( V, \tilde{A} \)

can we say \( \| e^{-itH} P_c(H) u_0 \|_{L_t^p L_x^q} \leq \| u_0 \|_2 \)?

Decay as \( x \to \infty \)?

Local regularity?
Thm: Suppose $V \in L^\infty$ and $\lim_{|x| \to \infty} |x|^2 V(x) = 0$.

$\vec{A}$ is continuous, and $\sum_{k=0}^{\infty} 2^k \sup_{|x|=2^k} |\vec{A}(x)|^2 < \infty$.

and $H$ does not have an eigenvalue or a resonance at $\lambda = 0$.

Then $\| e^{-itH} P_{ac}(H) u_0 \|_{L_t^p L_x^q} \leq \| u_0 \|_2$, $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, $p > 2$.

Remarks:

- Valid in dimensions $n \geq 3$
- Lose $p=2$ endpoint because of Christ-Kiselev lemma.
- Natural scaling is $V_0 \propto |x|^{-2}$
  $\vec{A}_\infty \propto |x|^{-1}$
- Probably true for $V_0 \in L^2(R^n)$
- Can't commute derivatives in $\left( \vec{A} \cdot \nabla + \nabla \cdot \vec{A} \right)$. 
Method of Proof (Rodnianski-Schlag, '04):

Use Duhamel's solution formula.

\[ u(t,x) = e^{it\Delta}u_0(t,x) - i \int_0^t e^{i(t-s)\Delta}L u(s,x) \, ds \]

First term is already good.
Now factorize \( L = \sum_{j=1}^J Y_j^* Z_j \).

That leaves \( \int_0^t e^{it\Delta}(e^{-is\Delta}Y_j^*)(Z_j e^{-isH}u_0) \, ds \).

We would like to have \( \| Z_j e^{-isH}P_{ac}(H) u_0 \|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq \| u_0 \|_2 \)
and \( \| \int_0^\infty e^{-is\Delta}Y_j^* q(s,x) \, ds \|_2 \leq \| q \|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \).

Finally, \( e^{it\Delta} \) maps \( L^2(\mathbb{R}^n) \) to \( L_t^p L_x^q \).

\[ \text{[Need Christ-Kiselev lemma because domain is } 0 \leq s \leq t \text{]} \]
Evaluating $\| Y_j e^{is\Delta} f \|_{L^2(\mathbb R^\times \mathbb R^n)}^2$

$\mathcal F^* + \text{Plancherel's Identity (} s \leftrightarrow \lambda \text{)}$

$\| Y_j e^{is\Delta} f \|_{L^2}^2 = \sup_{\lambda \in [0,\infty]} \| Y_j (R_o^+(\lambda) - R_o^-(\lambda)) Y_j^* \|_{L^2 \to L^2}$

where $R_o^\pm(\lambda) = \lim_{\varepsilon \to 0} \left( -\Delta - (\lambda \pm i\varepsilon) \right)^{-\frac{1}{2}}$.

- Fourier Multiplier $\frac{1}{|x|^2 - \lambda} \mp \frac{\pi i}{\sqrt{\lambda}} \text{ d}x$ $|x| = \sqrt{\lambda}$

- Convolution with Kernel

$K(\lambda, x) = |x|^{-n+2} K(\sqrt{\lambda} x) \sim \left\{ \begin{array}{ll} |x|^{-n+2} & \text{if } \sqrt{\lambda} |x| \leq 1 \\
\frac{\pi i |x|}{|x| \sqrt{\lambda}} & \text{if } \sqrt{\lambda} |x| > 1. \end{array} \right.$

Facts: $R_o(\lambda)$ maps $\langle x \rangle^{\frac{n+2}{2}} L^2(\mathbb R^n)$ to $\langle x \rangle^{\frac{n+2}{2}} L^2(\mathbb R^n)$

With operator norm $\sim \lambda^{\frac{n}{2}}$.

Powers of $\nabla$ increase norm by a factor of $\lambda^{\frac{n}{2}}$.

Typical $Y_j$ are: $\langle x \rangle^{\frac{n+2}{2}} |\nabla|^{\frac{n}{2}}$

or $\langle x \rangle^{-\frac{n}{2}-\varepsilon}$
Analysis of $\mathbf{Z}_j$ is the same, except we use $R_L^+(\lambda) = (H - (\lambda \pm i0))^{-1}$

$$= R_o(\lambda) \left[ I + LR_o(\lambda) \right]^{-1}$$

uniformly bounded?

For $\lambda \in [0, \lambda_1]$, use Fredholm Alternative (Agmon, '75).

**1st Problem:** $L$ is first-order.

Operator norm of $\nabla R_o(\lambda)$ does not decay in the limit $\lambda \to \infty$.

Uniform control of $[I + LR_o(\lambda)]^{-1}$ via power series?

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**Thm (Erdogan - G - Schlag '06 - '07):**

Given any $\varepsilon > 0$, there exists $m_\varepsilon < \infty$ so that

$$\limsup_{\lambda \to \infty} \left\| (LR_o(\lambda))^m \right\|_{x \to x} < \varepsilon^m.

(i.e. spectral radius vanishes as $\lambda \to \infty$)
Idea of Proof:

\((LR_\alpha(\lambda))^m\) is an \(m\)-fold integral operator.

Its kernel oscillates like \(e^{\pm i\sqrt{\lambda}(|x_0-x_1|+|x_1-x_2|+\cdots+|x_{m-1}-x_m|)}\)

There is stationary phase whenever \(x_0, x_1, x_2, \ldots, x_m\) are colinear and ordered.

Eventually \(|x_k|\) is large enough that \(A(x_k)\) is small.

(cf. Volterra operators)

Since \(A(x)\) isn't smooth, we can't use IBP in the complementary region.

Riemann-Lebesgue lemma is good enough.
2nd Problem: How to factorize $\hat{A}(x) \cdot \nabla = Y_j^* Z_j$

If both $Y_j, Z_j$ are (function of $x$)(1/2 derivative)
then you need control of $|1^{1/2} A(x) 1^{1/2}|$.

$\Rightarrow \quad A \in W^{1/2,2n} \cap L^\infty$.

New Idea:

\[
Y_j = \langle a_t^{1/4} \rangle \langle x \rangle^{-1/4}
\]

\[
Z_j = \langle a_t^{-1/4} \rangle \langle x \rangle \hat{A}(x) \cdot \nabla
\]

This works because $\langle a_t \rangle^3$ behaves like $\langle D_x \rangle^{2/3}$
but it commutes with $\hat{A}(x)$.

Minor Complication: Return to Duhamel formula

\[
\int_0^t e^{i(s-s)A} \mathcal{P}_c(H) u_0 \mathrm{d}s
\]

Domain of integration: Not supported in $\{s \geq 0\}$

is not $\{t \geq s \geq 0\}$

Because Fractional Derivatives
Are Non-Local.