Directions: This is a pencil-and-paper exam. You are asked to put away all books, notes, calculators, cell phones, and other computing and/or telecommunications equipment. The last page of this booklet is blank and is intended for use as scrap paper. Additional sheets of paper are available upon request.

Grading: Each numbered question is worth 10 points, for a total of 100 points.

1. Find the solution of the equation \( y'(x) + \frac{3x^2 + 2}{2y(x)} = 0 \) which goes through the point \((1, -1)\).

Find the values of \( y(-2) \) and \( y(2) \), if they exist.
2a. Find all solutions to the differential equation $y''(t) - 6y'(t) + 10y(t) = 0$.

2b. Find all solutions to the differential equation $y''(t) - 6y'(t) + 10y(t) = 6e^{2t}$ which have $y(0) = 1$. 
3a. Consider the second-order linear differential equation \((t - 1)y''(t) - ty'(t) + y = 0\), which is solved by both \(y_1(t) = t\) and \(y_2(t) = e^t\).

What is the Wronskian \(W[y_1, y_2](t)\)?
Is this a fundamental set of solutions at time \(t = 0\)?

3b. Find the solution to this equation with initial conditions \(y(0) = 1\) and \(y'(0) = 1 - e\).

What is unusual about this solution at \(t = 1\), and why isn’t it a violation of the Existence and Uniqueness Theorem?
4a. Consider the linear system of equations
\[
\begin{align*}
  x'_1(t) &= 3x_1(t) \\
  x'_2(t) &= -2x_1(t) + x_2(t)
\end{align*}
\]
What kind of critical point is there at the origin?
What are the eigenvalues and eigenvectors?

4b. Sketch the phase portrait for this system on the axes provided.
Problems 5 - 6 will concern the autonomous system of equations  
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix} 
-2 & 1 \\
0 & -1
\end{bmatrix}.
\]

5a. Find the critical points of this system.
For each critical point, write down the matrix of the associated linear system.

5b. Classify each of the critical points above [as a node, saddle point, spiral point, or center] and indicate whether it is stable.
We are still considering the system of equations \[
\begin{bmatrix}
x' \\
y
\end{bmatrix} = \begin{bmatrix} y \\ x - x^2 - xy \end{bmatrix}.
\]

6a. It is known that every function of the form
\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = Ce^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]
is a solution. Draw these trajectories on the axes below.

Also use your answers to Problem 5 to sketch phase portraits near each of the critical points.

\[\begin{array}{|c|c|}
\hline
& \\
\hline
-2 & -1 \\
\hline
1 & 2 \\
\hline
-1 & -2 \\
\hline
\end{array}\]

6b. Let \(\vec{x}(t)\) be the solution which satisfies the initial condition \(\vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}\).

What is \(\vec{x}'(0)\)? What is \(\lim_{t \to \infty} \vec{x}(t)\)?

Sketch the trajectory of \(\vec{x}(t)\) on the axes provided.
7a. Suppose an autonomous system of equations
\[
\begin{align*}
    x'(t) &= f(x(t), y(t)) \\
    y'(t) &= g(x(t), y(t))
\end{align*}
\]
has the property that \( x f(x, y) + y g(x, y) < 0 \) at every point except the origin.
Show that for every solution \([x(t), y(t)]\), the quantity \( \|\vec{z}(t)\|^2 = [x(t)]^2 + [y(t)]^2 \) is always decreasing.

7b. Explain why the origin must be a stable critical point,
and why there cannot be any other critical points or limit cycles.
8a. Suppose $f(x)$ is represented by the power series $\sum_{n=0}^{\infty} a_n x^n$.

Write out the power series representations for $f'(x)$ and $xf(x)$.

8b. Find a power series solution for the differential equation $f'(x) = xf(x)$.

What is the pattern for the coefficients $a_n$?

Do not solve the differential equation using other methods first – that would be cheating...
9a. Suppose we wanted to rewrite the function \( f(x) = \begin{cases} \cos x, & \text{if } |x| < \frac{\pi}{2} \\ 0, & \text{if } \frac{\pi}{2} \leq |x| \leq \pi \end{cases} \), with \( f(x + 2\pi) = f(x) \), as the Fourier series \( f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \).

Use the Euler-Fourier formulas to express each of the coefficients \( a_n \) and \( b_n \) as a definite integral, then evaluate \( a_0 \) exactly.

9b. True or false: \( b_n = 0 \) for every \( n \).

If it is true, give a brief explanation why.
If it is false, find a specific value of \( n \) for which \( b_n \neq 0 \).
10a. The wave equation
\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \]
has solutions in the form \( u_n(x, t) = \sin \left( \frac{n\pi}{L} x \right) T_n(t) \).

What differential equation must \( T_n(t) \) solve?
Based on this, what is the general form of the function \( T_n(t) \)?

10b. If a function \( u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) \) is a solution to the wave equation,
what relationship exists between \( u(x, t + 2L) \) and \( u(x, t) \)?