1 Introduction

Algebraic $K$-theory is the meeting ground for various other subjects such as algebraic geometry, number theory, and algebraic topology. We will try to expose some of these connections. Due to the historical development of algebraic $K$-theory, the field has two main components: the study of lower $K$-groups (in particular the Grothendieck group $K_0$), which have explicit algebraic descriptions used in applications, and the study of higher $K$-theory, which requires much more sophisticated categorical and homotopical machinery. However, it is the more general categorical approach that allows for the means of proving the fundamental theorems that make $K$-theory into a field of study in its own right. We give a broad introduction to both lower and higher $K$-groups with a focus on Quillen’s “$Q=Plus$” theorem which proves that two different definitions of the $K$-groups, both very useful in different contexts, are the same: one definition uses the plus construction of the classifying space of the group $GL(R)$ for a ring $R$, and the other uses Quillen’s $Q$-construction for an exact category. Then we present what is considered to be the most basic of the fundamental theorems in $K$-theory, the additivity theorem.

2 Lower $K$ groups

2.1 $K_0$ of a ring

The idea behind defining the zeroth $K$ groups was to strengthen the structure of an abelian monoid by formally adding inverses and thus forming its group completion. The most basic example is that of the monoid $\mathbb{N}$ whose group completion is $\mathbb{Z}$. In general, for an abelian monoid $M$ we define the Grothendieck group of $M$, denoted $\text{Gr}(M)$, to be the free abelian group on generators $[m]$ for $m \in M$, modulo the subgroup generated by elements of the form $[m] + [n] - [m+n]$, and define the group completion map $M \rightarrow \text{Gr}(M)$ to be the map that sends each $m$ to the class $[m]$. This map has the universal property that every monoid homomorphism from $M$ to an abelian group factors through it.

The $K_0$ groups were introduced by Grothendieck around 1956 in the context of sheaves over algebraic varieties. Inspired by Grothendieck’s work, M. Atiyah and F. Hirzebruch introduced topological $K$-theory in 1959. They define the $K_0$ group of a finite CW complex $X$ as the Grothendieck group of an abelian monoid $\text{Vect}(X)$ of isomorphism classes of finite-dimensional vector bundles (real or complex) over $X$ with the Whitney sum of vector bundle operation:

$$KU^0(X) = \text{Gr}(\text{Vect}_C(X)) \text{ and } KO^0(X) = \text{Gr}(\text{Vect}_R(X)).$$

This motivates the following algebraic definition of the group $K_0(R)$, where $R$ is a ring with unity.

**Definition 1.** Let $\text{P}(R)$ denote the abelian monoid of isomorphism classes of finitely generated projective $R$-modules. Then $K_0(R)$ is the Grothendieck group $\text{Gr}(\text{P}(R))$.

Why do we restrict attention to finitely generated projective modules? If $P \oplus Q = R^n$, then

$$P \oplus R^\infty \cong P (\oplus Q \oplus P) \oplus (Q \oplus P) \oplus \ldots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \ldots \cong R^\infty.$$ 

This “Eilenberg swindle” would imply that $[P] = 0$ for every projective $R$-module, and that $K_0(R) = 0$. 

Example 1. Let $F$ be a field. Then every finitely generated module over $F$ is free, so the abelian monoid $\mathbf{P}(F)$ is isomorphic to $\mathbb{N}$ and $K_0(F) = \mathbb{Z}$. More generally, if $R$ is a PID, the structure theorem for modules over a PID implies that every projective module is free, so again we get $K_0(R) = \mathbb{Z}$.

Example 2. Another example, which shows the relevance of the $K_0$ group in number theory, is $K_0(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the number ring of a number field $K$. Recall that a number ring is always a Dedekind domain, and the classification of finitely generated projective modules over a Dedekind ring tells us that for every finitely generated projective module $P$ of rank $n$, $P \cong \mathcal{O}_K^{-1} \oplus I$, where $I$ is a uniquely determined class in the ideal class group of $K$. Thus we get $K_0(\mathcal{O}_K) = \mathbb{Z} \oplus \text{Cl}(K)$.

Example 3. To give an example which has an application to topology, we consider the $K_0$ group for an integral group ring $\mathbb{Z}[\pi]$. Suppose $X$ is a path-connected space with the homotopy type of a CW complex and with fundamental group $\pi$, and that $X$ is a retract of a finite CW complex. Let $K_0(\mathbb{Z}[\pi])$ be the cokernel of the natural homomorphism $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}[\pi])$. Then there is an element in $K_0(\mathbb{Z}[\pi])$, the Wall finiteness obstruction, which vanishes if and only if $X$ is homotopy equivalent to a finite CW complex.

Example 4. Another example, which connects the definitions of the algebraic $K_0$ group and the topological $K^0$ group, is the following result of Swan. Consider the ring $\mathcal{C}(X, \mathbb{C})$ of continuous functions $X \to \mathbb{C}$ on a compact Hausdorff space $X$, and let $\eta : E \to X$ be a complex vector bundle. Then the set $\Gamma(E) = \{ s : X \to E : \eta s = 1_X \}$ of global sections of $\eta$ forms a projective $\mathcal{C}(X, \mathbb{C})$-module. Swan’s theorem asserts that actually the category of complex vector bundles over $X$ is equivalent to the category $\mathbf{P}(\mathcal{C}(X, \mathbb{C}))$, so we obtain an isomorphism $KU^0_{\text{top}}(X) \cong K_0(\mathcal{C}(X, \mathbb{C}))$. Similarly, we obtain $KO^0_{\text{top}}(X) \cong K_0(\mathcal{C}(X, \mathbb{R}))$.

### 2.2 $K_1$ of a ring

Let $R$ be a ring with unity. We can embed the group $GL_n(R)$ into $GL_{n+1}(R)$ by identifying an $n \times n$ matrix $A$ with $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. We define the infinite general linear group $GL(R)$ to be the union of the resulting sequence of inclusions.

**Definition 2.** $K_1(R)$ is the abelian group $GL(R)/[GL(R), GL(R)]$.

So $K_1(R)$ has the universal property that every homomorphism from $GL(R)$ to an abelian group factors through the quotient $GL(R) \to K_1(R)$. Whitehead proved that the commutator subgroup $[GL(R), GL(R)] = E(R)$, where $E(R)$ is the normal subgroup generated by the elementary matrices, i.e., the matrices with exactly one non-zero nondiagonal entry and all diagonal entries equal to 1. Thus we get $K_1(R) = GL(R)/E(R)$.

If $R$ is commutative, the determinant homomorphism $\det : GL(R) \to R^\times$ induces a homomorphism

$$\det : K_1(R) \to R^\times$$

whose kernel we denote by $SK_1(R)$. This homomorphism is split by the inclusion of $R^\times$ as $GL_1(R)$ in $GL(R)$.

**Example 5.** If $F$ is a field, standard linear algebra shows that a matrix of determinant 1 over $F$ is a product of elementary matrices; thus we get $E(R) = SL(R)$. So $K_1(F) = F^\times$.

**Example 6.** If $\mathcal{O}_K$ is the ring of integers of a number field $K$, then by a theorem of Bass-Milnor-Serre we have that $SK_1(\mathcal{O}_K) = 0$. Thus

$$K_1(\mathcal{O}_K) \cong \mathcal{O}_K^\times.$$

This result is complemented by the Dirichlet Theorem which says that

$$\mathcal{O}_K^\times = \mu(K) \oplus \mathbb{Z}^{r_1 + r_2 - 1}$$

where $\mu(K)$ is the unit group of $K$ and $r_1$ and $r_2$ denote the number of real, respectively the number of pairs of conjugate complex, embeddings of $K$ in $\mathbb{C}$.

**Example 7.** We give an application of $K_1$ to topology. We define the Whitehead group for a group $\pi$ as

$$Wh(\pi) = K_1(\mathbb{Z}[\pi]) / \langle \pm g \mid g \in \pi \rangle$$

where the elements of $\pi$ are regarded as elements of $GL_1(\mathbb{Z}[\pi])$. Two finite CW complexes have the same simple homotopy type if they are connected by a finite sequence of “elementary expansions and collapses.” Suppose $f : X \to Y$ is a homotopy equivalence of finite CW complexes with fundamental group $\pi$. Whitehead proved that there is an element $\tau(f) \in Wh(\pi)$, the torsion of $f$, such that $\tau(f) = 0$ iff $f$ is a simple homotopy equivalence, and that every element of $Wh(\pi)$ is the torsion of some homotopy equivalence $f$. 

2
3 Higher K theory

3.1 Topological K-theory

First, we will explain how all the \( K \)-groups are defined in topological \( K \)-theory, in order to emphasize how the two theories start to differ, higher algebraic \( K \)-groups being much harder to get a handle on. Note that as topological groups \( \text{GL}(\mathbb{R}) \simeq O \) and \( \text{GL}(\mathbb{C}) \simeq U \), where \( O \) and \( U \) denote the infinite unions of orthogonal, and unitary groups, respectively, along the usual inclusions. Also, \( BO \) and \( BU \) will stand for the classifying spaces of \( O \) and \( U \). We have the following representability theorem:

**Theorem 8.** For a finite CW complex \( X \), we have

\[
KO^0(X) = [X, BO \times \mathbb{Z}] \quad \text{and} \quad KU^0(X) = [X, BU \times \mathbb{Z}].
\]

Now Bott periodicity asserts that there are homotopy equivalences

\[
BO \times \mathbb{Z} \simeq \Omega^8(BO \times \mathbb{Z}) \quad \text{and} \quad BU \times \mathbb{Z} \simeq \Omega^2(BU \times \mathbb{Z}),
\]

and that the period 8 periodic homotopy groups \( \pi_i(BO \times \mathbb{Z}) \) are \( \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0 \), and the period 2 periodic homotopy groups \( \pi_i(BU \times \mathbb{Z}) \) are \( \mathbb{Z} \) and \( 0 \).

Thus \( BO \times \mathbb{Z} \) and \( BU \times \mathbb{Z} \) define \( \Omega \)-spectra \( KO \) and \( KU \), by which we mean sequences of spaces with the homotopy type of CW complexes \( \{E_0, E_1, \ldots\} \) such that \( \Omega E_i \simeq E_{i-1} \) for all \( i > 0 \). A spectrum gives rise to a generalized cohomology theory that satisfies all the Steenrod-Eilenberg axioms except the dimension axiom, with the cohomology groups defined as \( E^n(X) = [X, E_n] \) for \( n \geq 0 \). We define the topological \( K \)-theories \( KO^* \) and \( KU^* \) to be the generalized cohomology theories given by the spectra \( KO \) and \( KU \). So the complex \( K \)-groups will be periodic of period 2, and the real ones will be periodic of period 8.

3.2 The Plus-Construction

By analogy with topological \( K \)-theory, one might try to define higher algebraic \( K \)-groups for a ring \( R \) using the space \( BGL(R) \), where again, \( BGL(R) \) stands for the classifying space of the group \( GL(R) \). However, now \( GL(R) \) is a discrete group, so \( \pi_1(BGL(R)) = GL(R) \) and all other homotopy groups are 0, as opposed to the periodic homotopy groups of \( BO \) and \( BU \) when \( O \) and \( U \) were topological groups. This suggests one needs to make a modification to the construction, and Quillen defined the higher \( K \)-groups by using the Quillen \(+\)-construction of \( BGL(R) \): \( K_1(R) = \pi_1(BGL(R)^+) \) for \( i \geq 1 \), where the plus-construction is taken with respect to the commutator subgroup of \( GL(R) \), which can be easily shown to be a perfect subgroup using the Whitehead lemma. Recall that the plus-construction kills a perfect subgroup of the fundamental group and is an isomorphism on homology. In particular,

\[
K_1(R) = \pi_1(BGL(R)^+) = \pi_1(BGL(R))/[GL(R), GL(R)] = GL(R)/[GL(R), GL(R)],
\]

so our definition is consistent with our previous definition of \( K_1(R) \). We can define the \( K \)-theory space \( K(R) \) as the product \( K_0(R) \times BGL(R)^+ \). Then, by construction, \( K_i(R) = \pi_i(K(R)) \). This construction is related to group completions in the sense that one way to obtain the space \( BGL(R)^+ \) is as the basepoint component of the “group completion” of the \( H \)-space \( \coprod_{n=0}^{\infty} BGL_n(R) \). Recall that a homotopy associative, homotopy commutative \( H \)-space \( X \) is a topological space with a multiplication that is associative and commutative up to homotopy, and an identity element in the sense that multiplication by this element is homotopic to the identity. Its set of path components \( \pi_0(X) \) is an abelian monoid, \( H_0(X; \mathbb{Z}) \) is the monoid ring \( \mathbb{Z}[\pi_0(X)] \), and the integral homology \( H_{*}(X; \mathbb{Z}) \) is an associative graded-commutative ring with unit. Here, what we mean by group completion is the following, and the motivation behind this construction comes from the group completion of an abelian monoid discussed above:

**Definition 3.** A group completion of a homotopy associative, homotopy commutative \( H \)-space \( X \) is a homotopy associative, homotopy commutative \( H \)-space \( Y \), together with an \( H \)-space map \( X \to Y \), such that \( \pi_0(Y) \) is the group completion of the abelian monoid \( \pi_0(X) \), and the homology ring \( H_*(Y; R) \) is isomorphic to the localization \( \pi_0(X)^{-1}H_*(X, R) \) for all commutative rings \( R \).

Quillen’s motivation behind this definition of higher \( K \)-groups was to compute the \( K \)-groups of finite fields, which he was able to do by describing \( BGL(R)^+ \) as the homotopy fiber of a computable map.
3.3 The \( Q \)-construction

An exact category is a category with a notion of short exact sequences. We omit the definition since it is lengthy, in favor of some examples that should give the reader some intuition. An abelian category is exact, and thus the category of finitely generated left \( R \)-modules over a left Noetherian ring \( R \) is exact. Any additive category is exact by taking the exact sequences to be the split exact sequences, and so the category \( \mathbf{P}(R) \) of finitely generated projective modules over a ring \( R \) is exact with the exact sequences being the split exact ones.

If \( C \) is an exact category, we call a monomorphism (resp. epimorphism) appearing in a short exact sequence an admissible monomorphism (resp. epimorphism), and denote these by \( A \twoheadrightarrow B \) (resp. \( B \rightarrowtail C \)).

**Definition 4.** For an exact category \( C \) we define the category \( QC \); the objects are the same as the objects of \( C \) and a morphism from \( A \) to \( B \) is an isomorphism class of diagrams in \( C \) of the type

\[
A \xleftarrow{c} C \xrightarrow{d} B.
\]

Before we give the definition of the \( K \)-groups of a category we must introduce the notion of classifying space of a category. Given a small category \( C \), we can form the nerve of \( C \), a simplicial set with \( n \)-simplices being diagrams of the form \( c_0 \rightarrow c_1 \rightarrow \cdots \rightarrow c_n \) in \( C \). We define the \( i^{\text{th}} \) face map by replacing \( c_{i-1} \rightarrow c_i \rightarrow c_{i+1} \) by the composition \( c_{i-1} \rightarrow c_{i+1} \) for \( 0 < i < n \), we define the 0\(^{\text{th}} \) face map by deleting \( c_0 \rightarrow \) and the last face map by deleting \( \rightarrow c_n \). We define the \( i^{\text{th}} \) degeneracy map by replacing \( c_i \) by \( c_i \xrightarrow{e} c_i \). Thus we can take the geometric realization of this simplicial set and obtain a CW complex \( BC \). Also, note that functors between categories induce maps of the classifying spaces.

The data of a natural transformation \( \eta : F_0 \Rightarrow F_1 \) between two functors \( F_i : C \rightarrow D \) determines the data of a homotopy \( BC \times [0,1] \rightarrow BD \) between the maps \( BF_0 \) and \( BF_1 \). Thus any adjoint pair of functors induces a homotopy equivalence of categories, and therefore any category with an initial or terminal object is contractible. We can define homotopy groups, homology, etc., of categories by taking the corresponding construction for the classifying space of the category. In view of this homotopy theoretic approach, Quillen gave the following definition of \( K \)-groups:

**Definition 5.** For an exact category \( C \), we define the \( K \)-theory space by \( K(C) = \Omega BQC \). The \( K \)-groups are then defined by

\[
K_i(C) = \pi_i(K(C)) = \pi_{i+1}(BQC).
\]

Using this definition, Quillen was able to generalize the fundamental theorems that had been proved for \( K_0 \) and \( K_1 \) such as the additivity, the resolution, the devissage, and the localization theorem, which for brevity we will not state here.

3.4 Plus=Q Theorem

The goal is now to prove that the two above definitions agree, namely:

**Theorem 9 (Plus=Q).** If \( R \) is a ring, then

\[
\Omega B(Q\mathbf{P}(R)) \simeq BGL(R)^+ \times K_0(R).
\]

The strategy of the proof will be to define an intermediate object, the category \( S^{-1}S \) associated to a certain symmetric monoidal category \( S \), and to show that \( B(S^{-1}S) \) is homotopy equivalent to each of the terms in the equivalence we are trying to prove. Given a symmetric monoidal category \( S \) (i.e., a category with an operation \( \oplus : S \times S \rightarrow S \) which is commutative, associate and with a distinguished identity element, all up to coherent natural isomorphism), such as the categories \( \mathbf{M}(R) \) and \( \mathbf{P}(R) \) of finitely generated and finitely generated projective modules over \( R \) with the operation \( \oplus \), then \( BS \) is a homotopy commutative, homotopy associative \( H \)-space with multiplication induced by \( \oplus \). In order to avoid dealing with categories with an initial or terminal object, which are contractible, we restrict our attention to the category of isomorphisms \( \text{iso}S \). This is still symmetric monoidal, so \( B(\text{iso}S) \) is an \( H \)-space. We can generalize the notion of an abelian monoid acting on a set to that of a symmetric monoidal category acting on a category, and we can define the translation category:

**Definition 6.** If \( S \) acts on a category \( X \), we define the category \( \langle S, X \rangle \): it has the same objects as \( X \) and a morphism from \( x \) to \( y \) in \( \langle S, X \rangle \) is an equivalence class of pairs \( (s, s \oplus x \xrightarrow{\phi} y) \), with \( s \in S \) and \( \phi \) a morphism in \( X \) (where, by abuse of notation, \( \oplus \) denotes the action). Two pairs \( (s, \phi) \) and \( (s', \phi') \) are equivalent if there is an isomorphism \( s \cong s' \) making the relevant diagram commute.
We write $S^{-1}X$ for $(S, S \times X)$, and note that $S$ acts on $S^{-1}X$ by $s \oplus (t, x) = (s \oplus t, x)$. Moreover, it acts invertibly in the sense that the translation $(t, x) \mapsto (s \oplus t, x)$ has an homotopy inverse the translation $(t, x) \mapsto (t, s \oplus x)$ (since there is a natural transformation $(t, x) \mapsto (s \oplus t, s \oplus x)$).

**Proof sketch of Plus= $S^{-1}$.**

We want to show that for $S = \text{isoP}(R)$, we have $B(S^{-1}S) \simeq K_0(R) \times BGL(R)^+$. Note that in general $S^{-1}S$ is symmetric monoidal with $(m_1, n_1) \oplus (m_2, n_2) = (m_1 + m_2, n_1 \oplus n_2)$, and we have a monoidal functor $S \to S^{-1}S$ given by $s \mapsto (s, 0)$, where 0 is the unit of $S$. This induces an $H$-space map $BS \to B(S^{-1}S)$, and thus a map of abelian monoids

$$\pi_0(BS) \to \pi_0(B(S^{-1}S)).$$

But $\pi_0(B(S^{-1}S))$ is an abelian group (since $(s, t)$ can be seen to be the inverse of $(t, s)$) and this map is in fact a group completion. Since $BS$ is an $H$-space, $\pi_0S$ is a multiplicatively closed subset of the ring $H_0(S) = \mathbb{Z}[\pi_0S]$, so it acts on $H_*(S)$ and it acts invertibly on $H_*(S^{-1}S)$. Thus we get an induced map

$$(\pi_0S)^{-1}H_*(S) \to H_*(S^{-1}S).$$

We show that if $S = \text{isoS}$ and translations are faithful, i.e., $\text{Aut}(s) \to \text{Aut}(s \oplus t)$ is injective for all $s, t \in S$, then the induced $H$-space map $BS \to B(S^{-1}S)$ is a group completion, and in order to prove this claim it remains to show that the above map is an isomorphism.

It turns out that the projection functor $p : S^{-1}S \to (S, S)$ is cofibered (which means that the inverse image of each object is isomorphic to a certain type of category associated to it, called the comma category), and this will imply that there is an associated spectral sequence with $E_{pq}^2 = H_p((S, S); H_q(S)) \Rightarrow H_{p+q}(S^{-1}S)$. We can localize at $\pi_0(S)$, which acts invertibly on $H_*(S^{-1}S)$, so we get a spectral sequence $E_{pq}^2 = H_p((S, S); (\pi_0S)^{-1}H_q(S)) \Rightarrow H_{p+q}(S^{-1}S)$ that will degenerate to the isomorphism we want.

Now let $T = \text{isoF}(R)$, where $F(R)$ denotes the category of free $R$-modules. So in $T$ the set of morphisms $R^n \to R^m$ is $GL_n(R)$ if $n = m$ and it is empty otherwise. Thus $B(T) = \coprod_n BGL_n(R)$. As a general fact which is not hard to see, $X \simeq \pi_0(X) \times X_0$, if $X$ is a homotopy associative and commutative $H$-space with $\pi_0(X)$ a group, and where $X_0$ denotes the basepoint component of $X$. Using the mapping telescope from the construction of $BGL(R)$, we can construct a map $BGL(R) \to B_0(T^{-1}T)$ and because $B(T^{-1}T)$ is a group completion of $B(T)$, this map will be acyclic, and this gives us $BGL(R)^+ \simeq B_0(T^{-1}T)$. Thus we get

$$B(T^{-1}T) \simeq \mathbb{Z} \times BGL(R)^+.$$

Now, for a cofinal functor (i.e., a functor $F : T \to T$ such that for every $s \in S$ there is $s' \in S$ and $t \in T$ such that $s \oplus s' = F(t)$) if $\text{Aut}_T(t) \cong \text{Aut}_S(F(t))$ for all $t \in T$, we can show that the map $B(T^{-1}T) \to B(S^{-1}S)$ induces a homotopy equivalence of basepoint components. Note that the inclusion functor $F(R) \to \text{P}(R)$ is cofinal since every projective module is a direct summand of a free module. Combining this with our previous result, we get that for $S = \text{isoP}(R)$,

$$B(S^{-1}S) \simeq K_0(R) \times BGL(R)^+.$$

**Proof sketch of $S^{-1}S = Q$.**

We seek to show an equivalence $\Omega BQC \simeq B(S^{-1}S)$, where $C$ is a split exact category and $S = \text{isoC}$. The strategy will be to fit $B(S^{-1}S)$ and $B(QC)$ into an appropriate homotopy fibration sequence with contractible term in the middle.

**Definition 7.** For an exact category $C$, we define a category $\mathcal{EC}$ as follows. The objects of $\mathcal{EC}$ are the exact sequences in $C$. A morphism from $A \rightarrow B \rightarrow C$ to $A' \rightarrow B' \rightarrow C'$ is an equivalence class of diagrams

$$\begin{array}{ccc}
A & \rightarrow & B & \rightarrow & C \\
\alpha \downarrow & & \downarrow & & \downarrow \\
A' & \rightarrow & B' & \rightarrow & C'
\end{array}$$

where two such diagrams are equivalent if there is an isomorphism between them which is the identity except at $C''$. We can define a functor $T : \mathcal{EC} \to QC$ by $T(A \rightarrow B \rightarrow C) = C$ (since the right column in the above diagram is just a morphism in $QC$ from $C'$ to $C$). We will write $\mathcal{T}_C$ for $T^{-1}(C)$. Also, note that $S$ acts on $\mathcal{EC}$ by

$$(A \rightarrow B \rightarrow C) \mapsto (S \oplus A \rightarrow S \oplus B \rightarrow C),$$

...
and since $T(S \oplus (A \to B \to C)) = T(A \to B \to C),$ we get an induced functor $T : S^{-1}EC \to QC.$ Suppose $C$ is split exact. Define a functor $S \to EC$ by $A \mapsto (A \to A \oplus C \to C).$ Then the induced map $B(S^{-1}S) \to B(T^{-1}(C)) = B(S^{-1}EC)$ is an equivalence. Now if $S = isoC,$ we can use Quillen’s theorem B (which gives a criterion for a sequence to be a homotopy fibration) to show that

$$B(S^{-1}S) \to B(S^{-1}EC) \xrightarrow{BT} B(QC)$$

is a homotopy fibration. Then, it remains to show that $B(S^{-1}EC)$ is contractible in order to conclude that $\Omega BQC \simeq B(S^{-1}S).

4 The Additivity Theorem

A Waldhausen category $(C, coC, wC)$ is a category with appropriate notions of cofibrations (which form a subcategory $coC$) and weak equivalences (which form a subcategory $wC$). Again, we omit the lengthy precise definition, but mention some examples. The pointed category of finite based sets with injective functions as cofibrations and isomorphisms as weak equivalences forms a Waldhausen category. Any exact category becomes a Waldhausen category with admissible monics as cofibrations and isomorphism as weak equivalences. Another example is the category of finite based CW complexes with cofibrations being the cellular inclusions and weak equivalences being the weak homotopy equivalences. Waldhausen’s initial motivation was to have a construction which satisfies the “cylinder axiom”, analogous to the formation of the mapping cylinder.

We construct the $K$-theory space of a Waldhausen category $C$ the following way. First, we define a simplicial category $S.C$ with $n$-simplices $S_nC$ being categories whose objects are sequences of $n$ cofibrations in $C$:

$$0 = A_0 \to A_1 \to \cdots \to A_n$$

with compatible choices of subquotients $A_{ij} = A_j/A_i$ for all $0 < i < j \leq n$, in the sense that they make some $n \times n$ upper triangular diagram commute. By deleting the $i^{th}$ row and column of this diagram we can define face maps, and by duplicating the $A_i$ we can define degeneracy maps, which all fit together to form a simplicial Waldhausen category. The subcategories $wS_nC$ fit together to form a simplicial category $wS.C.$ We let

$$K(C) = \Omega BwS.C,$$

where $BwS.C$ is the geometric realization of the simplicial category $wS.C.$ Note that since $S_0C$ is trivial, $BwS.C$ is a connected space, so we do not lose any homotopical information by passing to the loop space. The $K$-groups of $C$ are defined as the homotopy groups of $K(C)$:

$$K_i(C) = \pi_i K(C) = \pi_{i+1} BwS.C.$$

Of course, if $C$ is just an exact category regarded as a Waldhausen category in the way described above, the definition of the $K$-groups agrees with the previous one, since $S.C$ turns out to be naturally homotopy equivalent to Quillen’s category $QC.$

We state the additivity theorem, which was initially introduced by Quillen for exact categories, in the more general setting of Waldhausen categories.

**Theorem 10.** The exact functor $(s, q) : EC \to C \times C$ defined by $A \mapsto B \mapsto C \mapsto (A, C)$ induces a homotopy equivalence $K(EC) \xrightarrow{\sim} K(C) \times K(C).$

Heuristically, this means that $K$-theory takes into account the splitting of exact sequences. The statement is equivalent to the assertion that for an appropriate notion of an exact sequence of functors $F' \to F \to F'' : C \to D$ between Waldhausen categories $C$ and $D,$ $F$ and $F' \vee F''$ induce homotopic maps between the $K$-theory spaces $K(C)$ and $K(D).$ Waldhausen’s initial proof of the additivity theorem uses simplicial versions of Quillen’s theorems A and B (which give criteria for a functor to be a homotopy equivalence, and for a sequence to be a homotopy fibration, respectively). There have been several simplifications and alternate proofs since, the newest one by Daniel Grayson constructing an explicit combinatorial homotopy by subdividing the spaces involved.

The importance of the additivity theorem is emphasized in [7], where Staffeldt uses it to reprove the other fundamental theorems of $K$-theory. He claims in the introduction that “the additivity theorem is promoted to the status of the most basic theorem in algebraic $K$-theory.”
References


[5] May, Peter *Notes on Dedekind rings*


[10] Guillou, Bertrand *Algebraic K-Theory*