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EQUIVARIANT ALGEBRAIC $K$-THEORY

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For my father, in loving memory.
I was the shadow of the waxwing slain
By the false azure in the windowpane
I was the smudge of ashen fluff—and I
Lived on, flew on, in the reflected sky

(Nabokov, Pale Fire)
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ABSTRACT

It is a common phenomenon in mathematics to understand objects (such as vector spaces, number fields, topological spaces) through inherent group actions on them. However, spectra with $G$-action (naive $G$-spectra), are not robust enough for stable homotopy theory, and the objects of study in equivariant stable homotopy theory are genuine $G$-spectra, which correspond to $RO(G)$-graded cohomology theories. Experience has proven that the powerful tools of genuine equivariant homotopy theory can shed light on nonequivariant problems.

We give a systematic study of rings with actions by a group $G$ and we construct and compare several $G$-categories of modules over such $G$-rings. When $G$ is finite, we show how to output corresponding genuine $K$-theory $G$-spectra from such $G$-categories. This makes the tools of equivariant stable homotopy theory directly available for the study of $K$-theory. Central to the development of equivariant algebraic $K$-theory is finding models for classifying spaces of equivariant bundles. We show how to interpret equivariant topological real and complex $K$-theory, and Atiyah’s Real $K$-theory from this perspective.

We prove that the $G$-fixed point spectrum of a Galois extension of rings is the nonequivariant algebraic $K$-theory spectrum of the $G$-fixed ring. Statements previously formulated in terms of naive $G$-spectra for Galois extensions, such as the Quillen-Lichtenbaum conjecture and Carlsson’s conjecture can be formulated in terms of genuine $G$-spectra. Lastly, we encode an action on exact and Waldhausen categories and define an equivariant algebraic $K$-theory space for these, which we conjecture to be an infinite loop $G$-space. As a first application in this direction, we define the space-level equivariant algebraic $K$-theory $A_G(M)$ of a $G$-manifold $M$. 

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1.1 Naive and genuine equivariant theories

A $G$-action on an object in a category can be thought of as a group homomorphism from $G$ to the group of automorphisms of that object. Whereas equivariant homotopy theory is the systematic homotopical study of group actions on spaces, equivariant stable homotopy theory is more intricate than just the study of group actions on spectra. Spectra are sequences of spaces related by suspension maps, and they are the representing objects for generalized cohomology theories in the sense that cohomology groups of a space are given by homotopy classes of maps into the spaces of a spectrum. Spectra are “stable” analogues of spaces: we can suspend and desuspend spectra by spheres of any dimension. A group action on a spectrum gives rise to the notion of a naive $G$-spectrum. Unfortunately, the theory one obtains from naive spectra does not allow for the usual duality phenomena from stable homotopy theory. For example, we do not even get Poincaré duality for $G$-manifolds. In order to rectify this, one needs to allow suspension by representation spheres $S^V$, which are one point compactifications of representations $V$ of the group $G$. This leads to the notion of a genuine $G$-spectrum, which is indexed on representations of $G$ and in turn represents a cohomology theory graded on representations. The philosophy of this work is to play the two notions off of each other, namely we start with naive $G$-actions and encode them as genuine $G$-spectra.

Equivariant stable homotopy theory is a very fascinating subject in its own right, but what has really driven its development is the fact that the powerful tools of genuine equivariant homotopy theory can shed light on nonequivariant problems. An early example of such an application is Carlsson’s proof of the Segal conjecture, which states that the zeroth stable cohomotopy group of the classifying space of a group is isomorphic to a completion of
the Burnside ring of the group. The proof of this nonequivariant statement requires heavy use of equivariant theory (see [Car84]). A more recent example is the Hill-Hopkins-Ravenel solution to the Kervaire invariant one problem, which is ultimately a statement about manifolds – the proof, however, unexpectedly relies on sophisticated equivariant machinery (see [HHR]). The solution to the Kervaire invariant one problem has reinvigorated interest in equivariant homotopy theory, since it demonstrates the power of equivariant techniques for solving problems in nonequivariant homotopy theory.

1.2 Algebraic $K$-theory

Algebraic $K$-theory is the meeting ground for various other subjects such as algebraic geometry, number theory, and algebraic topology. Lower $K$-groups have explicit algebraic descriptions and encode geometric-topological information such as Wall’s finiteness obstruction and Whitehead torsion. Higher algebraic $K$-groups require sophisticated topological and categorical machinery to define, and their introduction by Quillen was the culminating point of a long search for a definition that would meaningfully generalize the existing definitions of lower $K$-groups. Despite computations being very hard, algebraic $K$-groups are intensely studied because they encode very deep number theoretic information. For example, algebraic $K$-groups of number rings can be used to formulate a generalization of the class number formula, and knowledge of the $K$-theory of $\mathbb{Z}$ would imply the Vandiver conjecture.

The idea behind defining the zeroth $K$ group was to strengthen the structure of an abelian monoid $M$ by formally adding inverses and thus forming its group completion $Gr(M)$. Let $\mathcal{P}(R)$ be the category of finitely generated projective modules and let $iso \mathcal{P}(R)$ be the groupoid of isomorphisms. Then $\pi_0(iso \mathcal{P}(R))$, the set of isomorphism classes of objects, is an abelian monoid with sum defined by $[P] + [Q] = [P \oplus Q]$ – however, it is not a group because there are no $R$-modules that have negative rank. The group $K_0(R)$ is defined as the
Higher algebraic $K$-theory generalizes this idea. Starting with a category $\mathcal{C}$, Grothendieck defined its classifying space $B\mathcal{C}$, which at least for groupoids, preserves the essential information. The intuitive picture is that one draws a point for each object of $\mathcal{C}$, paths between these points representing the arrows in the category, then one fills in a triangle for every pair of composable morphisms with boundary given by the edges of the two maps and their composition, and continuing in this fashion for higher composable morphisms. The precise definition is in two steps: one forms a simplicial set, the nerve of $\mathcal{C}$ with $n$-simplices the $n$-composable morphisms and then one takes the geometric realization. Note that for a groupoid $\mathcal{C}$, the set of isomorphism classes of objects, $\pi_0(\mathcal{C})$, is the same as $\pi_0(B\mathcal{C})$.

The idea of Quillen’s higher algebraic $K$-theory is to strengthen the structure of a topological (as opposed to just an algebraic) monoid, such as the classifying space $B\mathcal{C}$ of a symmetric monoidal groupoid $\mathcal{C}$, by topologically adjoining homotopy inverses. This is made precise in the notion of topological group completion of a homotopy commutative and homotopy associative $H$-space. The map

$$B\mathcal{C} \longrightarrow K(\mathcal{C})$$

is defined as such a group completion. (For $R$ a ring, one takes $\mathcal{C} = \text{iso } \mathcal{P}(R)$.) This is compatible with the algebraic construction:

$$K(\pi_0(\mathcal{C})) \cong \pi_0(K(\mathcal{C})).$$

Amazingly, it turns out that the group completion of $B\mathcal{C}$ for $\mathcal{C}$ a symmetric monoidal category is not just a homotopy associative and homotopy multiplicative $H$-space, but its
multiplication is actually coherent up to all higher homotopies, which makes it into an \textit{infinite loop space}, so there is a sequence of spaces $X_i$ such that

$$K(\mathcal{C}) \simeq \Omega X_1 \simeq \Omega^2 X_2 \simeq \cdots \Omega^n X_n \simeq \cdots.$$ 

Thus $\{X_i\}$ represents an $\Omega$-spectrum and hence a cohomology theory. Therefore $K(\mathcal{C})$ bears much more structure than $B\mathcal{C}$. The procedure for obtaining the deloopings of $K(\mathcal{C})$, namely the associated $\Omega$ spectrum whose 0th space it represents, is to use an infinite loop space machine such as the operadic one developed by May [May72] or the $\Gamma$-space one developed by Segal [Seg74].

If $\mathcal{C}$ is an exact category, or more generally, a Waldhausen category (i.e. a category with cofibrations and weak equivalences), then $K_0(\mathcal{C})$ is defined by imposing the relation $[A] + [B] = [C]$ not only when $A \oplus B \simeq C$, but also for every exact sequence $0 \to A \to C \to B \to 0$. In the same spirit, Quillen’s $Q$-construction for exact categories and Waldhausen’s $S_\bullet$-construction for Waldhausen categories are used to construct a $K$-theory space, which is not just the group completion of a monoid structure anymore, but it has the property that it \textit{splits exact sequences}, which can be made precise via the \textit{additivity theorem}. Again, these spaces turn out to be infinite loop spaces, but in order to deloop them, since infinite loop space machines with the group completion property are not suited to deal with the extra structure of the exact sequences, the method is to iterate the constructions to obtain the deloopings.$^1$

The homotopy groups of these $K$-theory spaces or spectra have great formal properties; they satisfy the \textit{fundamental theorems}. One example of these is the localization theorem,

$^1$ For the $Q$ construction, one cannot actually just iterate because $Q\mathcal{C}$ is not an exact category anymore and a more complicated iteration process is necessary. However, $S_\bullet \mathcal{C}$ yields a simplicial Waldhausen category so that the construction can be iterated, and the iterations yield the deloopings.
which allows reduction of many computations of $K$-groups to the calculation of the $K$-theory of fields. It has long been understood that the $K$-theory of a field should be computable in terms of the $K$-theory of the algebraic closure and the action of the absolute Galois group. One of the versions of the Quillen-Lichtenbaum conjectures makes this precise in terms of a map involving naive $G$-spectra.

1.3 Equivariant algebraic $K$-theory

This thesis is focused on encoding an inherent action on the input to create a genuine equivariant version of algebraic $K$-theory, which makes the tools of equivariant stable homotopy theory directly available for the study of $K$-theory. At the same time, we connect equivariant topological $K$-theory, Atiyah’s Real $K$-theory, which is an analogue of topological $K$-theory for Real vector bundles, and existing $K$-theory statements about naive $G$-spectra, such as the statement of the Quillen-Lichtenbaum conjecture, in a more conceptual way under one unifying framework, and interpret them from the perspective of full-fledged equivariant stable homotopy theory.

1.3.1 Categorical homotopy fixed points

If $\mathcal{C}$ and $\mathcal{D}$ are $G$-categories, let $\text{Cat}(\mathcal{C}, \mathcal{D})$ denote the functor category, where $G$ acts on functors and natural transformations by conjugation. For a group $G$, let $\tilde{G}$ denote the category with objects the elements of $G$ and a unique morphism between any two objects. The group $G$ acts by translation on the objects and diagonally on the morphisms, thus $\tilde{G}$ is $G$-isomorphic to the translation category of $G$.

If $\mathcal{C}$ is a $G$-category with certain structure (e.g. symmetric monoidal, exact, Waldhausen), then the functor category $\text{Cat}(\tilde{G}, \mathcal{C})$ also has the same structure and it is nonequivariantly equivalent to $\mathcal{C}$ (but this is not necessarily a $G$-equivalence). By analogy with the homotopy fixed point set of a $G$-space, we think of the fixed points of $\text{Cat}(\tilde{G}, \mathcal{C})$ as the
**categorical homotopy fixed points** of \(\mathcal{C}\), following Thomason [Tho83].

In Chapter 2, we give an analysis of homotopy fixed point categories, since these will be at the heart of the definition of equivariant algebraic \(K\)-theory. For instance, a module category over a \(G\)-ring \(R\) has an action of \(G\) by twisting the scalar multiplication of the modules. This \(G\)-category does not have interesting fixed points: the fixed points are just modules over the fixed subring. However, the homotopy fixed points turn out to be the modules over the twisted group ring, which we denote by \(R_G[G]\), and therefore they capture the \(G\)-action on the ring.

Our philosophy, however, is to work with the equivariant object \(\text{Cat}(\tilde{G}, \mathcal{C})\), as opposed to just restricting attention to its fixed points. This object encodes interesting information at its \(H\)-fixed points, which are the \(H\)-homotopy fixed points of \(\mathcal{C}\). For example, for the module category of a \(G\)-ring \(R\), \(\text{Cat}(\tilde{G}, \text{Mod}(R))\) has as \(H\)-fixed points the category of modules over the twisted group ring \(R_H[H]\).

A wonderful feature of working with the category \(\text{Cat}(\tilde{G}, \mathcal{C})\) instead of \(\mathcal{C}\) is that a \(G\)-map \(\mathcal{C} \to \mathcal{D}\) between \(G\)-categories, which is a nonequivariant equivalence, induces an equivalence of homotopy fixed points, and thus a weak \(G\)-equivalence

\[
\text{Cat}(\tilde{G}, \mathcal{C}) \to \text{Cat}(\tilde{G}, \mathcal{D}).
\]

We very much exploit this phenomenon. For instance, the inclusion \(\bigsqcup_n GL_n(R) \hookrightarrow \text{iso } \mathcal{F}(R)\) of the skeleton of the category of free \(R\)-modules and isomorphisms is not a \(G\)-map. However, the inverse equivalence turns out to be equivariant when \(G\) acts on the skeleton by fixing all objects and entrywise on matrices. Therefore, we have a weak \(G\)-equivalence after applying the functor \(\text{Cat}(\tilde{G}, -)\). This will allow us to use the skeleton in algebraic \(K\)-theory without losing any equivariant information.

There are interesting maps between \(G\)-categories, which are not \(G\)-maps, but for which the equivariance holds up to isomorphism. We introduce the concept of pseudo equivariant
functors, and show that these induce honest $G$-maps after applying the functor $\text{Cat}(\widetilde{G}, -)$. We use the concept of pseudo equivariance to give a definition of equivariant Morita equivalence that equivariant algebraic $K$-theory will be invariant under.

1.3.2 Equivariant bundles and classifying spaces

Quillen’s definition of higher algebraic $K$-groups was as the homotopy groups of the group completion of the topological monoid $B(\coprod_n GL_n(R)) = \coprod_n BGL_n(R)$. The idea for the equivariant algebraic $K$-theory groups is to replace the classifying spaces of principal $GL_n(R)$-bundles by classifying spaces of equivariant bundles, which we wish to understand as classifying spaces of categories. This is carried out in Chapter 3, which is joint work with B. Guillou and J.P. May.

Let $G$ be a topological group acting on another topological group $\Pi$, so that we have a semi-direct product $\Gamma = \Pi \rtimes G$ and a split extension

$$1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1.$$ (1.1)

There is a general theory of bundles corresponding to such extensions, which we refer to as $(G, \Pi_G)$-bundles, and which is particularly well-known for $\Gamma = O(n) \rtimes G$ or $U(n) \rtimes G$ - these yield equivariant topological $K$-theory. It turns out that the case $\Gamma = \Sigma_n \rtimes G$ is relevant to infinite loop space theory and of course, the case $\Gamma = GL_n(R) \rtimes G$, with $G$ acting entrywise on $GL_n(R)$ is central to equivariant algebraic $K$-theory.

We find the following surprisingly simple model for the classifying space of $(G, \Pi_G)$-bundles:

**Theorem 1.3.1** (Theorem 3.3.8). If $G$ is discrete and $\Pi$ is discrete or compact Lie, the canonical map

$$B\text{Cat}(\widetilde{G}, \widetilde{\Pi}) \rightarrow B\text{Cat}(\widetilde{G}, \Pi),$$

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is a universal principal $(G, \Pi_G)$-bundle.

The fixed point category of $\text{Cat}(\tilde{G}, \Pi)$ also has a surprising description: the set of isomorphism classes of objects in this category is the first nonabelian cohomology set $H^1(G, \Pi)$. This identification is relevant for the equivariant algebraic $K$-theory of Galois extensions, when Hilbert’s theorem 90 says that the first nonabelian cohomology is trivial.

### 1.3.3 Equivariant algebraic $K$-theory of $G$-rings

In Chapter 4, we construct a functor $K_G$ from $G$-rings to genuine $G$-spectra. This is built in two steps.

The first step is to define the equivariant algebraic $K$-theory space of a $G$-ring. In Definitions 4.2.1 and 4.2.2, we define the equivariant algebraic $K$-theory space $K_G(R)$ as the equivariant group completion of the classifying space of the symmetric monoidal $G$-category $\text{Cat}(\tilde{G}, \mathcal{P}(R))$, where $\mathcal{P}(R)$ is the $G$-category of finitely generated projective $R$-modules.

The second step is to equivariantly deloop this $G$-space and obtain a genuine $\Omega$-$G$-spectrum $K_G(R)$ with zeroth space $K_G(R)$. For this we appeal to equivariant infinite loop space machines, such as the operadic one developed by Guillou-May in [GMa] or the Segalic one developed by [Shi89], which given appropriate categorical input produce genuine orthogonal $\Omega$-$G$-spectra. The definition of the $K$-theory space was rigged so that it is an infinite loop $G$-space, i.e., it has deloopings with respect to all finite dimensional representations of $G$. The equivalence of these equivariant infinite loop space machines, which allows their interchangeable role in algebraic $K$-theory, is shown in [MMO].

The $K$-theory of the twisted group rings $R_H[H]$ is built into our definition of equivariant algebraic $K$-theory of a $G$-ring $R$ for all subgroups $H \subseteq G$:

**Theorem 1.3.2** (Theorem 4.4.9). If $H \subseteq G$ and $|H|^{-1} \in R$, there is an equivalence of spectra

$$K_G(R)^H \simeq K(R_H[H]).$$
We define the equivariant algebraic $K$-groups $K^H_i(R)$ to be the equivariant homotopy groups of the $G$-space $K_G(R)$, or equivalently of the spectrum $K_G(R)$. Therefore, the functors $K_i(R)$ from the orbit category of $G$ to the category of abelian groups, defined by

$$K_i(R)(G/H) = K^H_i(R),$$

are Mackey functors for all $i$.

**Definition 1.3.3** (Definition 4.2.4). The algebraic $K$-theory groups are defined as

$$K^H_i(R) = \pi^H_i(K_G(R)).$$

We show that the higher equivariant algebraic $K$-groups are the equivariant homotopy groups of the equivariant group completion of the topological $G$-monoid of classifying spaces of $(G, GL_n(R))$-bundles with operation given by Whitney sum. In particular, this shows that our definition agrees with the one given by Fiedorowicz-Haushild-May in [FHM82] for rings with trivial $G$-action. By Theorem 1.3.1, the topological monoid of classifying spaces of $(G, GL_n(R))$-bundles is $BCat(\tilde{G}, \mathcal{G}\mathcal{L}(R))$, where $\mathcal{G}\mathcal{L}(R) = \bigsqcup GL_n(R)$. A model for the group completion is $\Omega BB\text{Cat}(\tilde{G}, \mathcal{G}\mathcal{L}(R))$.

**Theorem 1.3.4** (Theorem 4.3.3). For $i > 0$, we have an isomorphism of equivariant homotopy groups

$$K^H_i(R) \cong \pi^H_i(\Omega BB\text{Cat}(\tilde{G}, \mathcal{G}\mathcal{L}(R))).$$

If $R = F$ is a field, then we have an isomorphism on all equivariant homotopy groups, including $\pi_0$, and therefore we have a weak $G$-equivalence

$$K_G(F) \simeq \Omega BB\text{Cat}(\tilde{G}, \mathcal{G}\mathcal{L}(F)).$$

This is useful because the construction $\Omega BB\text{Cat}(\tilde{G}, \mathcal{G}\mathcal{L}(F))$ works for topological $G$-fields.
$F$, as a result of the fact that the structure groups for the equivariant bundles from Theorem 1.3.1 are allowed to be topological. The following theorems, showing that our definition recovers the known equivariant topological $K$-theories, fall out of our definition of equivariant algebraic $K$-theory. Equivariant topological real and complex $K$-theory $K U_G$ and $K O_G$ were defined by Segal in [Seg68] and the $K$-theory of Real vector bundles $K R$ was introduced by Atiyah in [Ati66]. We denote by $k u_G$, $k o_G$ and $k r$ the connective versions.

**Theorem 1.3.5** (Theorem 4.7.1, Theorem 4.7.2). *Consider the topological rings $\mathbb{C}$ and $\mathbb{R}$ with trivial $G$-action for any finite group $G$, and the topological ring $\mathbb{C}^{\text{conj}}$ with $C_2$-conjugation action. We have equivalences of connective $\Omega$-$G$-spectra*

$$K_G(\mathbb{C}) \simeq k u_G, \ K_G(\mathbb{R}) \simeq k o_G, \ K_{C_2}(\mathbb{C}^{\text{conj}}) \simeq k r.$$  

Another very desirable property of equivariant algebraic $K$-theory is the following theorem regarding Galois extensions of rings, an example of which are the usual Galois extensions of fields. We give an exposition of ring Galois extensions in Section 4.5.3.

**Theorem 1.3.6** (Theorem 4.5.11). *Let $R \rightarrow S$ be a Galois extension of rings with finite Galois group $G$. Then we have an equivalence of orthogonal spectra*

$$K_G(S)^G \simeq K(R).$$

In particular, this recovers $K(\mathbb{Q})$ as the fixed point spectrum of the genuine equivariant $K$-theory spectrum of any Galois extension of $\mathbb{Q}$.

In Section 4.5.6, we give an interpretation of the map in the Quillen-Lichtenbaum conjecture in terms of genuine $G$-spectra. Let $E/F$ be a field Galois extension, the initial Quillen-Lichtenbaum conjecture was that the map of spectra

$$K F = (K E)^G \longrightarrow K E^{hG} \tag{1.2}$$
is an equivalence after $p$-completion, where $(KE)^G$ and $KE^{hG}$ denote the fixed points and the homotopy fixed points of the naive spectrum $KE$. Thomason, however, showed that this map only becomes an equivalence after reducing modulo a prime power and inverting the Bott element. We show that the map (1.2) is equivalent to the map

$$KF \simeq K_G(E)^G \longrightarrow K_G(E)^{hG}$$

from fixed points to homotopy fixed points of the genuine $G$-spectrum $K_G(E)$.

There has been a long standing program initiated and lead by G. Carlsson of studying the $K$-theory of fields motivated by the concept of descent and the Quillen-Lichtenbaum conjecture. The goal of Carlsson’s program is finding a spectrum, which depends on the absolute Galois group and whose homotopy groups are the $p$-completed homotopy groups of $KF$. Carlsson’s conjecture states that the representational assembly map from the $K$-theory spectrum of the symmetric monoidal category of continuous $k$-representations of the absolute Galois group of $F$ to the $K$-theory spectrum of $F$ is an equivalence after suitable completion, where $k \subseteq F$ is an algebraically closed field.

In Sections 4.6.1 and 4.6.2 we give an overview of Carlsson’s conjecture and for finite Galois extensions, we interpret the map that figures in the conjecture as the fixed points of an equivariant map of genuine equivariant $K$-theory spectra. It is not hard to see that the spectra involved in the representational assembly map, which Carlsson calls “equivariant $K$-theory spectra” in [Car] actually arise as $G$-fixed point spectra of our construction of genuine equivariant $K$-theory spectra. What is a little less obvious is how to get an equivariant map, which induces the representational assembly map on fixed points. This uses the notion of pseudo equivariant functor between $G$-categories developed in Section 2.4.
1.3.4 Equivariant algebraic $K$-theory of exact and Waldhausen categories

In [DK82], Dress and Kuku independently define equivariant algebraic $K$-groups for exact and Waldhausen categories with trivial $G$-action. The cases of interest to us will be categories which have a nontrivial $G$-action, and we want the definition of equivariant $K$-theory to see the action. In Chapter 5, we give a definition of the equivariant algebraic $K$-theory space of an exact or a Waldhausen $G$-category, which generalizes Kuku’s definition.

Note that if $\mathcal{C}$ is an exact or Waldhausen $G$-category, then $\mathcal{C}(\tilde{G}, \mathcal{C})$ is also an exact or Waldhausen $G$-category. Our definition of the equivariant algebraic $K$-theory space is as follows.

**Definition 1.3.7** (Definition 5.2.5, Definition 5.4.3). The equivariant algebraic $K$-theory space of $\mathcal{C}$ is $\Omega QBCat(\tilde{G}, \mathcal{C})$ if $\mathcal{C}$ is an exact $G$-category and $\Omega |wS\cdot \text{Cat}(\tilde{G}, \text{iso} \mathcal{C})|$ if $\mathcal{C}$ is a Waldhausen $G$-category.

We show that this definition agrees with the previous definition given for a $G$-ring. The following can be regarded as a $plus = Q = S\cdot$ theorem.

**Theorem 1.3.8** (Theorem 5.3.1, Proposition 5.4.4). For a $G$-ring $R$, there are $G$-equivalences

$$K_G(R) \simeq \Omega QBCat(\tilde{G}, \mathcal{P}(R)) \simeq \Omega |wS\cdot \text{Cat}(\tilde{G}, \mathcal{P}(R))|.$$

For a $G$-space $X$, we define a $G$-action on the Waldhausen category of retractive spaces $R(X)$, and we define equivariant $A_G$-theory as the equivariant algebraic $K$-theory of this Waldhausen $G$-category.

We conjecture that the $K$-theory space we define is an infinite loop $G$-space, and we hope to be able to produce its deloopings in future work.

**Conjecture 1.3.9** (Conjecture 5.4.5). Let $\mathcal{C}$ be an exact or Waldhausen $G$-category. Then the equivariant $K$-theory space $K_G(\mathcal{C})$ is an infinite loop $G$-space.
It would be very interesting to study the equivariant algebraic $K$-theory spectrum of the category $R(X)$ when $X$ is a $G$-manifold, because this spectrum should split into the $G$-suspension spectrum of $X$ and a factor related to equivariant pseudoisotopies and $h$-cobordisms.
CHAPTER 2
HOMOTOPY FIXED POINT CATEGORIES

2.1 Homotopy fixed points for $G$-spaces

2.1.1 Homotopy fixed point space

We start by reviewing the definition of the homotopy fixed point space of a group action of $G$ on a space $X$. Let $EG$ be a universal principal $G$-bundle, i.e., a contractible space with free $G$-action. Then $\text{Map}(EG, X)$, the space of continuous maps $EG \to X$, inherits a $G$-action by conjugation $(gf)(y) = gf(g^{-1}y)$. If $X$ is a based space, the analogue is the space of based continuous maps $\text{Map}_*(EG_+, X)$, where $EG_+$ is the space $EG$ with a disjoint basepoint.

The homotopy fixed point space of $X$, denoted by $X^{hG}$ is defined as the $G$-fixed points $\text{Map}(EG, X)^G$, namely just the equivariant functions, which we denote by $G\text{Map}(EG, X)$. The trivially equivariant projection $EG \to *$ induces an equivariant map

$$X \longrightarrow \text{Map}(*, X) \longrightarrow \text{Map}(EG, X),$$

which is a nonequivariant equivalence since $EG$ is contractible. Thus nonequivariantly, $\text{Map}(EG, X)$ is equivalent to $X$. However, this is not necessarily a $G$-homotopy equivalence, since $EG \to *$ is not a $G$-equivalence: $EG^G = \emptyset$ whereas $^G = *$.

The homotopy fixed points $X^{hG}$ are sometimes easier to study than the actual fixed points $X^G$, because they are constructed in a homotopy theoretic way out of $BG$ and $X$. For example, if the $G$-action on $X$ is trivial, we get that

$$X^{hG} = G\text{Map}(EG, X) \cong \text{Map}(EG/G, X) = \text{Map}(BG, X).$$
In the general case, we get a spectral sequence

\[ E_2^{p,q} = H^p(G, \pi_q X) \implies \pi_{q-p}X^{hG}. \]

Note that since \( X \) is a space, ignoring the group action, \( \pi_q X \) is defined to be 0 for \( q < 0 \), it is just a set for \( q = 0 \), and it might not be abelian if \( q = 1 \). We can do the above construction in naive \( G \)-spectra and obtain an analogous homotopy limit spectral sequence. For \( X \) a spectrum \( \pi_q X \) is now an abelian group for all \( q \). Since spectra are cotensored over spaces, for a naive \( G \)-spectrum \( X \), we analogously define the homotopy fixed points \( X^{hG} \) as based maps \( \Map^* (E_G^+, X)_G^G \). If the spectrum \( X \) is given by \( \{X_i\} \), then levelwise the spectrum \( X^{hG} \) is given by \( \{X_i^{hG}\} \), i.e., the homotopy fixed points of the spaces that define the naive \( G \)-spectrum levelwise.

We also note that if \( X \) is a naive \( \Omega-G \)-spectrum, then \( X^{hG} \) is also an \( \Omega \)-spectrum. Indeed, the homeomorphisms

\[ \Map^* (E_G^+, \Omega X_i) \cong \Omega \Map^* (E_G^+, X_i) \]

yield a homeomorphism \( \Map^* (E_G^+, \Omega X_i)^G \cong \Omega \Map^* (E_G^+, X_i)^G \), since \( \Omega \) commutes with taking fixed points.

Remark 2.1.1. Let \( X \) be a \( G \)-space. Note that we have a weak equivalence

\[ \Map (E_G, X)^H \simeq \Map (E_H, X)^H, \]

since \( E_G \simeq_H E_H \) is an \( H \)-equivalence: indeed, both \( E_G \) and \( E_H \) are contractible free \( H \)-spaces for any subgroup \( H \subseteq G \). Thus we can define the \( H \)-homotopy fixed points of \( X \) either as the homotopy fixed points of \( X \) regarded as an \( H \)-space, or equivalently, as the \( H \)-fixed points of \( \Map (E_G, X) \).
2.1.2 Categorical interpretation of the homotopy fixed point space

We point out how the homotopy fixed points of a $G$-space can be interpreted as a categorical homotopy limit. Therefore, the notion of homotopy fixed points generalizes to categories of $G$-objects in model categories, which we describe in the following section.

Note that if $X$ is a right $G$-space, it can be regarded as a contravariant functor $X: G \to \text{Top}$, where $G$ is the category with a single object and endomorphism group $G$. Then

$$\lim_{G} X = X^{G} \quad \text{and} \quad \text{holim}_{G} X = X^{hG}.$$

Thus, the homotopy fixed points can be viewed as the derived fixed points. The same interpretation holds for the homotopy fixed points of a naive $G$-spectrum. This observation makes the following proposition clear, since homotopy limits are homotopy invariant. Alternatively, one could prove the proposition with more work directly from the definitions in the previous section.

Proposition 2.1.2. Let $X$ and $Y$ be $G$-spaces (naive $G$-spectra, resp.) and assume that $X \to Y$ is a $G$-map, which is a nonequivariant equivalence. Then $X^{hG} \to Y^{hG}$ is a weak equivalence.

Remark 2.1.3. Before we move on to the next section, we remark that, dually, we can define the homotopy orbits as $X_{hG} = EG \times_{G} X$. Note that the orbits of a left $G$-space $X$, which we regard as a covariant functor $X: G \to \text{Top}$ are given by $X/G = \text{colim}_{G} X$, and then $X_{hG} = \text{hocolim}_{G} X$. 

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2.2 \textit{G}-objects in categories

2.2.1 Definition and examples

Let $G$ be a group, which we regard as a category with one object, and let $\mathcal{C}$ be an arbitrary category.

**Definition 2.2.1.** A (left) $G$-\textit{object}, or $G$-\textit{representation} of $\mathcal{C}$, is a functor $G \to \mathcal{C}$.

Unpacking the definition, a $G$-object in $\mathcal{C}$ consists of the data of an object $X$ in $\mathcal{C}$ together with a group homomorphism $G \to \text{Aut}(X)$. We define $G\mathcal{C}$ to be the category of $G$-objects in $\mathcal{C}$ with morphisms being given by natural transformations. We give a few examples that illustrate that this definition captures the well-known notions of objects equipped with a group action.

**Example 2.2.2.** For $\mathcal{C} = \mathbb{F}$, where $\mathbb{F}$ is the category of finite sets, $G\mathcal{C}$ is the category of finite $G$-sets and $G$-maps.

**Example 2.2.3.** For $\mathcal{C} = \text{Mod}(R)$, the category of finitely generated modules over a ring $R$, $G\text{Mod}(R)$ is the category of finitely generated $R[G]$-modules and $G$-maps. In particular, if $R = F$ is a field, then $G\text{Mod}(F)$ is the category of $G$-representations over $F$, which we will also denote by $\text{Rep}_F[G]$.

**Example 2.2.4.** For $\mathcal{C} = \text{Top}$, the category of topological spaces, $G\text{Top}$ is the category of $G$-spaces, namely spaces with a continuous $G$-action, and $G$-maps.

**Example 2.2.5.** If $\mathcal{S}$ is the category of (pre)spectra, then $G\mathcal{S}$ is the category of \textit{naive} $G$-spectra, or spectra “with $G$-action:” The data of such a spectrum consists of a collection of $G$-spaces $\{X_i\}_{n \geq 0}$ and $G$-equivariant structure maps $\Sigma X_i \to X_{i+1}$, where $G$ acts trivially on the suspension coordinate. Similarly, we can get orthogonal or symmetric spectra with “$G$-action.”

**Example 2.2.6.** If $\mathcal{C} = \text{Cat}$, the category of categories and functors, then $G\text{Cat}$ is the category of $G$-categories and $G$-equivariant functors. We will analyze this example in detail.
2.2.2 Remarks on the homotopy theory of categories of $G$-objects

If $\mathcal{C}$ is a model category, there is a model structure on $G\mathcal{C}$ in which the fibrations and weak equivalences are the maps which are fibrations or weak equivalences, respectively, in $\mathcal{C}$. However, this model structure does not capture the “equivariant homotopy theory.” For example, in $G\text{Top}$, the right notion of a weak equivalence is not just a map of $G$-spaces $X \to Y$, which is a nonequivariant weak equivalence, but one which induces weak equivalences on fixed points $X^H \to Y^H$ for all $H \subseteq G$.

Similarly, if $\mathcal{S}$ is the category of (pre)spectra, then the desired notion of weak equivalence for spectra “with $G$-action” is not that of an underlying nonequivariant equivalence. Sometimes, spectra “with $G$-action” and this very raw notion of weak equivalence are referred to as coarse $G$-spectra. The desired notion of weak equivalence for naive $G$-spectra is given by the maps which induce a stable equivalence on fixed points, i.e., which induce isomorphisms on the equivariant homotopy groups defined as

$$\pi^H_q(X) = \colim_n \pi_{q+n}(X^H_n),$$

for all $H \subseteq G$.

We follow the outline given in [BMO$^+$] to explain the general principle of defining the weak equivalences in $G\mathcal{C}$. First, we can define a fixed point functor

$$(-)^H : G\mathcal{C} \to \mathcal{C}$$

by $X^H = \lim_H X$. Let Orb($G$) be the orbit category of $G$, i.e., the category with objects the orbits $G/H$ and $G$-maps. There is a map in Orb($G$) between $G/H$ and $G/K$ precisely when $K$ is sub conjugate to $H$, namely, when $gKg^{-1} \subseteq H$ for some $g \in G$. We define a functor

$$\Phi_X : \text{Orb}(G) \to \mathcal{C}$$
by \( G/H \mapsto X^H \).

As a matter of standard category theory, if \( \mathcal{C} \) is a cofibrantly generated model category, \( \text{Cat}(\text{Orb}(G), \mathcal{C}) \), the category of functors from \( \text{Orb}(G) \) to \( \mathcal{C} \), has a projective model structure, in which weak equivalences and fibrations are the natural transformations that are objectwise such morphisms in \( \mathcal{C} \).

The functor \( GC \longrightarrow \text{Cat}(\text{Orb}(G), \mathcal{C}) \) given by \( \mathcal{C} \mapsto \Phi \mathcal{C} \) has a left adjoint, and sometimes it can be used to lift the projective model structure from \( \text{Cat}(\text{Orb}(G), \mathcal{C}) \) to \( GC \). For example, that is how one gets the right notion of weak equivalence in \( G\text{Top} \). In [BMO+], the authors show that there is a model structure on \( G\text{Cat} \) such that a map \( \mathcal{C} \rightarrow \mathcal{D} \) is a weak equivalence or a fibration if \( \mathcal{C}^H \rightarrow \mathcal{D}^H \) is so for all \( H \subseteq G \) in the Thomason model structure on \( \text{Cat} \) (see [Tho80]). We note that, in particular, a functor is a weak equivalence in the Thomason model structure if it induces a weak equivalence after applying the classifying space functor. This is the notion of weak equivalence that we will use throughout this manuscript; we give the definition in 2.3.1.

### 2.3 Categorical homotopy fixed points

Now we specialize to \( G \)-categories, which we will use extensively throughout this work. In this section, we introduce the homotopy fixed points of a \( G \)-category, which we define by analogy with the homotopy fixed points for a \( G \)-space defined in section 2.1.1, and we explain how they fit into the conceptual setting from section 2.2.2. These were also studied by Thomason under the name “\( lax \) limit” in [Tho83]. However, we take an equivariant point of view: homotopy fixed points are the actual fixed points of a \( G \)-category, and we study this equivariant object as opposed to just restricting attention to the fixed points.
2.3.1 \( G \)-categories

From definition 2.2.1, a \( G \)-category is a functor \( G \to \text{Cat} \). We recapitulate that the data
of such a functor is a category \( \mathcal{C} \), and for each \( g \in G \), a functor \( (g\cdot) : \mathcal{C} \to \mathcal{C} \) such that
\( (e\cdot) = \text{id}_\mathcal{C} \) and \( (g\cdot) \circ (h\cdot) = (gh)\cdot \).

By slight abuse, we will often call the category \( \mathcal{C} \) a \( G \)-category, which means we are implicitly thinking of the action endofunctors \( (g\cdot) \). Sometimes we might omit the “\( \cdot \)” from the notation and write \( g\mathcal{C} \) or \( gf \) to denote the action of \( g \) on an object \( \mathcal{C} \) or a morphism \( f \). A natural transformation of functors \( G \to \text{Cat} \) translates to a functor between the two categories, which commutes with the \( G \)-action. Thus, as claimed in Example 2.2.6, \( \text{GCat} \) is the category of \( G \)-categories and \( G \)-equivariant functors.

We define the \( H \)-fixed point category \( \mathcal{C}^H \) of a \( G \)-category \( \mathcal{C} \) as the subcategory with
objects those \( \mathcal{C} \in \mathcal{C} \) such that \( h\mathcal{C} = \mathcal{C} \) and morphisms those \( f \in \mathcal{C} \) such that \( hf = f \) for all \( h \in H \). This definition coincides with the categorical definition as \( \lim_H \mathcal{C} \) when we think of \( \mathcal{C} \) as a functor \( G \to \text{Cat} \). The classifying space functor \( B : \text{Cat} \to \text{Top} \) commutes with fixed points, namely
\[
B(\mathcal{C}^H) = (B\mathcal{C})^H.
\]

**Definition 2.3.1.** A functor between \( G \)-categories \( F : \mathcal{C} \to \mathcal{D} \) is a weak \( G \)-equivalence if it induces a weak \( G \)-equivalence on classifying spaces \( BF : B\mathcal{C} \to B\mathcal{D} \).

These weak equivalences are lifted from the projective model structure on the category of functors \( \text{Orb}(G) \to \text{Cat} \), as mentioned in section 2.2.2.

Of course, by (2.1), if the maps \( \mathcal{C}^H \to \mathcal{D}^H \) are equivalences of categories, then the map \( \mathcal{C} \to \mathcal{D} \) is a weak \( G \)-equivalence. But it is enough for the maps \( \mathcal{C}^H \to \mathcal{D}^H \) to have adjoints, since adjoint functors induce an equivalence of categories upon application of the functor \( B \).
2.3.2 GCat as a 2-category

We may view Cat as the 2-category of categories, with 0-cells, 1-cells, and 2-cells the categories, functors, and natural transformations. From that point of view, Cat is enriched over itself: the internal hom, $\text{Cat}(\mathcal{A}, \mathcal{B})$, is the category whose objects are the functors $\mathcal{A} \to \mathcal{B}$ and whose morphisms are the natural transformations between them.

Similarly, we may view $G\text{Cat}$ as the underlying 2-category of a category enriched over $G\text{Cat}$. The 0-cells are $G$-categories, and the internal hom between them is the $G$-category $\text{Cat}(\mathcal{A}, \mathcal{B})$. Its underlying category is $\text{Cat}(\mathcal{A}, \mathcal{B})$, and $G$ acts by conjugation on functors and natural transformations. Thus, for $F: \mathcal{A} \to \mathcal{B}$, $g \in G$, and $A$ either an object or a morphism of $\mathcal{A}$, $(gF)(A) = gF(g^{-1}A)$. Similarly, for a natural transformation $\eta: E \to F$ and an object $A$ of $\mathcal{A}$,

$$(g\eta)_A = g\eta g^{-1}A: gE(g^{-1}A) \to gF(g^{-1}A).$$

The category $G\text{Cat}(\mathcal{A}, \mathcal{B})$ of $G$-equivariant functors and $G$-equivariant natural transformations is the same as the $G$-fixed category $\text{Cat}(\mathcal{A}, \mathcal{B})^G$.

Remark 2.3.2. We can topologize the definitions so far, starting with the 2-category of categories internal to the category Top, together with continuous functors and continuous natural transformations. A topological $G$-category $\mathcal{A}$ is a category internal to the cartesian monoidal category $G\text{Top}$. It has object and morphisms $G$-spaces and continuous $G$-equivariant source, target, identity and composition structure maps. These maps are denoted $S$, $T$, $I$, and $C$, and the usual category axioms must hold. These are more general than (small) topologically enriched categories, which have discrete sets of objects. We can now allow $G$ to be a topological group in the equivariant picture.
2.3.3 The functor $\text{Cat}(\tilde{G}, -)$

For a topological group $G$, define $\tilde{G}$ to be the topological $G$-groupoid with object space $G$ and morphism space $G \times G$. The source and target maps are the projections onto the two factors. Thus the objects of $\tilde{G}$ are the elements of $G$ and there is a unique morphism between any two objects. We choose to label the unique morphism $g \rightarrow h$ by the pair $(h, g)$ in order to be consistent with [GMM]. The idea is that reversing the order of source and target makes the notation for composition more transparent: $(g, h) \circ (h, k) = (g, k)$. The $G$-action on $\tilde{G}$ is given by translation on the objects, which forces it to be diagonal on morphisms, since $g(h \rightarrow k)$, namely $g(k, h)$ must be the unique map $gh \rightarrow gk$, namely $(gk, gh)$.

**Definition 2.3.3.** Define the translation category of $G$ of $G$ in the standard way as having object space $G$ and morphism space $G \times G$, with the morphism $h \rightarrow gh$ labeled by $(g, h)$.

Again, since there is a unique morphism between any two objects, the $G$-action on objects by translation completely determines the action on the morphism space: $G$ acts on the second coordinate of $G \times G$. The following lemma follows immediately from the fact that $G \times G$ with $G$ acting diagonally and $G \times G$ with $G$ acting on the second coordinate are $G$-homeomorphic.

**Lemma 2.3.4.** The translation category $G$ is $G$-isomorphic to the category $\tilde{G}$.

**Remark 2.3.5.** The category $\tilde{G}$ is an instance of the more general concept of chaotic category corresponding to a space. There is a chaotic category functor from spaces to categories (actually, to groupoids), sending a space $X$ to the category $\tilde{X}$ with object space $X$ and morphism space $X \times X$; there is a unique morphism between any two objects in $\tilde{X}$. The relevant point is that the object functor is right adjoint to the chaotic category functor, and in particular, we have a homeomorphism between the mapping spaces

$$\text{Cat}(\mathcal{C}, \tilde{X}) \cong \text{Map}(\text{Ob}\mathcal{C}, X).$$  \hspace{1cm} (2.2)
Similarly, the translation category $\tilde{G}$ of $G$ is an instance of the more general notion of translation category of a $G$-space. For a $G$-set, or a $G$-space $X$, we denote by $\mathcal{X}$ the translation category of $X$ with objects the points of $X$ and morphisms $(g, x) : x \to gx$. However, as we have seen, the concepts of chaotic and translation category agree for $G$ up to $G$-isomorphism. Thus, it is harmless to think of $\tilde{G}$ as the translation category of $G$. For a more comprehensive treatment of both chaotic and translation categories, we refer the reader to [GMM].

We make the following crucial observation.

*Observation* 2.3.6. Note that 

$$B\tilde{G} \simeq EG$$

since $\tilde{G}$ is a contractible category (every object is initial and terminal) and it has a free $G$-action.

We have a functor $\text{Cat}(\tilde{G}, -)$ from $G$-categories to $G$-categories, which sends a $G$-category $\mathcal{C}$ to the category of functors and natural transformations $\text{Cat}(\tilde{G}, \mathcal{C})$, with $G$-action by conjugation, as described in section 2.3.2. This is a topological category when $\mathcal{C}$ is such. In view of Section 2.3.2, $\text{Cat}(\tilde{G}, -)$ can be viewed as a 2-functor. Observe that the functor $\text{Cat}(\tilde{G}, -)$ is corepresented and is thus a right adjoint. Therefore it preserves all limits; in particular it preserves products, which will be crucial to our applications.

The equivariant projection $\tilde{G} \to *$ to the trivial $G$-category induces a natural $G$-map

$$\iota : \mathcal{A} \simeq \text{Cat}(*, \mathcal{A}) \to \text{Cat}(\tilde{G}, \mathcal{A}),$$

(2.3)

which is always a nonequivariant equivalence of $G$-categories, but not usually a $G$-equivalence.

However, the functor $\text{Cat}(\tilde{G}, -)$ is idempotent:
Lemma 2.3.7. For any $G$-category $\mathcal{A}$,

$$\iota: \text{Cat}(\tilde{G}, \mathcal{A}) \to \text{Cat}(\tilde{G}, \text{Cat}(\tilde{G}, \mathcal{A}))$$

is an equivalence of $G$-categories.

Proof. By adjunction, have a $G$-equivalence

$$\text{Cat}(\tilde{G}, \text{Cat}(\tilde{G}, \mathcal{A})) \simeq \text{Cat}(\tilde{G} \times \tilde{G}, \mathcal{A}).$$

Now $\iota$ is induced by the first projection $\tilde{G} \times \tilde{G} \to \tilde{G}$, which is a $G$-equivalence with inverse given by the diagonal. \qed

By analogy with the definition of homotopy fixed points of $G$-spaces, we make the following definition.

Definition 2.3.8. The homotopy fixed points of a $G$-category $\mathcal{C}$, denoted by $\mathcal{C}^{hG}$, are defined as $\text{Cat}(\tilde{G}, \mathcal{C})^G$, namely the $G$-equivariant functors $\tilde{G} \to \mathcal{C}$ and the $G$-natural transformations between these.

2.3.4 Explicit description of homotopy fixed point categories

We give an explicit description of the fixed point category $\text{Cat}(\tilde{G}, \mathcal{C})^H$. We have

$$\text{Cat}(\tilde{G}, \mathcal{C})^H = H\text{Cat}(\tilde{G}, \mathcal{C})) \simeq H\mathcal{C}at(\tilde{H}, \mathcal{C}); \quad (2.4)$$

the last equivalence follows because $\tilde{H}$ and $\tilde{G}$ are equivalent as $H$-categories since they are both $H$-free contractible categories.

So we have to understand the category of equivariant functors and equivariant natural transformations $G\mathcal{C}at(\tilde{G}, \mathcal{C})$. Any $H$-fixed functor $F: \tilde{G} \to \mathcal{C}$ is determined on objects
by where the identity $e$ of $G$ gets mapped to since $F(g) = g \cdot F(e)$. On morphisms, $F$ is determined by where it sends morphisms of the type $(g,e)$ since $F(g,h) = h \cdot F(h^{-1}g,e)$.

We have that $F(e,e) = id_C$, where $id_C$ is the identity morphism of the object $C \in \mathcal{C}$ and $F(e) = C$. The following cocycle condition is also satisfied:

$$F(gh,e) = F(gh,g)F(g,e) = g \cdot F(h,e)F(g,e).$$

We summarize this discussion in the following result, which gives an explicit description of the homotopy fixed point category of a $G$-category $\mathcal{C}$.\(^1\)

**Theorem 2.3.9.** The objects of the homotopy fixed point category $\mathcal{C}^{hG} = G\mathcal{C}at(\tilde{G}, \mathcal{C})$ are pairs $(C,f)$ where $C$ is an object of $\mathcal{C}$ and $f : G \rightarrow \text{Mor}(\mathcal{C})$ is a map from $G$ to morphisms of $\mathcal{C}$ such that $f(g) : C \rightarrow g \cdot C$ and $f$ satisfies the condition that $f(e) = id_C$ and the cocycle condition

$$f(gh) = (g \cdot f(h))f(g)$$

(2.5)

A morphism $(C,f) \rightarrow (C',f')$ is given by a morphism $\alpha : C \rightarrow C'$ in $\mathcal{C}$ such that the following diagram commutes for any $g \in G$:

$$F(gh,e) = F(gh,g)F(g,e) = g \cdot F(h,e)F(g,e).$$

However, the alternative cocycle condition

$$f(gh) = f(g)(g \cdot f(h))$$

(2.6)

is the standard one, which will appear in all of our applications. This is the condition that

\(^1\) The above explicit description is also given in more concise terms in [Tho85].

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yields a crossed homomorphism when \( \mathcal{C} \) is a group, as we will define in section 2.5, whereas condition (2.5) yields a crossed antihomomorphism, which is less customary.

We show that changing condition (2.5) to the usual cocycle condition (2.6) is inoffensive since it yields an equivalent category. We proceed to prove this.

**Theorem 2.3.10.** There is an isomorphism of categories between the homotopy fixed point category \( \mathcal{C}^hG = G\text{Cat}(\tilde{G},\mathcal{C}) \) and the category described as follows. The objects are pairs \((C,f)\) where \( C \) is an object of \( \mathcal{C} \) and \( f : G \to \text{Mor}(\mathcal{C}) \) is a map from \( G \) to morphisms of \( \mathcal{C} \) such that \( f(g) : g \cdot C \to C \) and \( f \) satisfies the condition that \( f(e) = \text{id}_C \) and the cocycle condition

\[
f(gh) = f(g)(g \cdot f(h)).
\]

A morphism \((C,f) \to (C',f')\) is given by a morphism \( \alpha : C \to C' \) in \( \mathcal{C} \) such that the following diagram commutes for any \( g \in G \):

\[
\begin{array}{ccc}
g \cdot C & \xrightarrow{f(g)} & C \\
g \cdot \alpha \downarrow & & \downarrow \alpha \\
g \cdot C' & \xrightarrow{f'(g)} & C'
\end{array}
\]

**Proof.** We explicitly construct the isomorphism between the category described in Theorem 2.3.10 to the category described in Theorem 2.3.9. The construction of the inverse isomorphism is similar.

Let \( f : G \to \text{Mor}(\mathcal{C}) \), such that \( f(g) : g \cdot C \to C \) and suppose that \( f \) satisfies \( f(e) = \text{id}_C \) and condition (2.6). Note that \( f(g) \) is an isomorphism with inverse \( g \cdot f(g^{-1}) \) for all \( g \in G \).

Define \( \tilde{f} : G \to \text{Mor}(\mathcal{C}) \) by

\[
\tilde{f}(g) = g \cdot f(g^{-1}),
\]
so that $\bar{f} : C \to g \cdot C$. Then

$$\bar{f}(gh) = (gh) \cdot f(h^{-1}g^{-1}) = g \cdot h \cdot (f(h^{-1})(h^{-1} \cdot f(g^{-1})) = (g \cdot \bar{f}(h))(\bar{f}(g)),$$

so that $\bar{f}$ satisfies condition (2.5).

Let $f, f' : G \to \text{Mor} (\mathcal{C})$, such that $f(g) : g \cdot C \to C$ and $f(g) : g \cdot C' \to C'$. Suppose that $f$ and $f'$ satisfy (2.6), and $\alpha : C \to C'$ is a morphism in $\mathcal{C}$ for which the diagram

\[
\begin{array}{ccc}
g \cdot C & \xrightarrow{f(g)} & C \\
g \cdot \alpha & \downarrow & \downarrow \alpha \\
g \cdot C' & \xrightarrow{f'(g)} & C'
\end{array}
\]

commutes for all $g \in G$. Then

$$\bar{f}'(g) \alpha = (g \cdot f'(g^{-1})) \alpha = g \cdot (f'(g^{-1})(g^{-1} \cdot \alpha)) = g \cdot (\alpha f(g^{-1})) = (g \cdot \alpha) \bar{f}(g),$$

i.e. the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\bar{f}(g)} & g \cdot C \\
\alpha & \downarrow & \downarrow g \cdot \alpha \\
C' & \xrightarrow{\bar{f}'(g)} & g \cdot C'
\end{array}
\]

commutes for all $g \in G$.

Remark 2.3.11. Note that if $G$ acts trivially on $\mathcal{C}$, then the category $G\mathcal{C}at(\tilde{G}, \mathcal{C})$ is just the category of functors $G \to \mathcal{C}$, i.e. it is the category of representations of $G$ in $\mathcal{C}$.

The following lemma is inspired by Proposition 2.1.2 for homotopy fixed points of $G$-spaces or naive $G$-spectra.

**Proposition 2.3.12.** If $\Theta : \mathcal{C} \to \mathcal{D}$ is a $G$-functor, which is a nonequivariant equivalence,
then the functor induced by post composition

$$\text{Cat}(\tilde{G}, \mathcal{C}) \rightarrow \text{Cat}(\tilde{G}, \mathcal{D})$$

is a weak $G$-equivalence, i.e., it induces equivalences

$$\text{Cat}(\tilde{G}, \mathcal{C})^H \rightarrow \text{Cat}(\tilde{G}, \mathcal{D})^H$$

for all $H \subseteq G$.

Proof. From our identification of $H$-fixed points made in (2.4), we see that it is enough to prove the result for $G$-fixed points. The map

$$\text{Cat}(\tilde{G}, \mathcal{C})^G \rightarrow \text{Cat}(\tilde{G}, \mathcal{D})^G$$

is faithful since it is the restriction of a faithful map to a subcategory. We show that is essentially surjective and full. We use the explicit description of fixed points given above.

Pick an object $(D, f)$ in $\text{Cat}(\tilde{G}, \mathcal{D})^G$. Since $\Theta$ is essentially surjective, there exists a nonequivariant isomorphism $\psi: D \xrightarrow{\cong} \Theta(C)$ for some $C \in \mathcal{C}$. By applying $g$, we get

$$g\psi: gD \xrightarrow{\cong} g\Theta(C) = \Theta(gC).$$

Since $\Theta$ is fully faithful for every $f(g): D \xrightarrow{\cong} gD$ there exists a unique map $f'(g): C \xrightarrow{\cong} gC$ such that $\Theta(f'(g))$ is the composite

$$\Theta(C) \xrightarrow{\psi^{-1}} D \xrightarrow{f(g)} gD \xrightarrow{g\psi} g\Theta(C),$$

and $f'(g)$ is an isomorphism since $f(g)$ and $\psi$ are. We need to check the cocycle condition on $f'$.
We will read it off the following commutative diagram

\[
\begin{array}{cccccc}
D & \xrightarrow{f(g)} & gD & \xrightarrow{g_f(h)} & ghD \\
\psi \downarrow & & \downarrow g\psi & & \downarrow g\psi \\
\Theta(C) & \xrightarrow{\Theta(f(g))} & g\Theta(C) & \xrightarrow{g\Theta(f(h))} & gh\Theta(C)
\end{array}
\]

The top composite is \( f(gh) \) since \( f \) satisfies the cocycle condition. Thus the bottom map must be \( \Theta(f(gh)) \). By the commutation of \( g \) with \( \Theta \), the bottom map is just \( \Theta \) applied to the composite

\[
C \xrightarrow{f'(g)} gC \xrightarrow{g(f'(h))} ghC.
\]

Thus \( f' \) satisfies the cocycle condition.

We are left to show fullness. Suppose we have a morphism in \( \text{Cat}(\tilde{G}, \mathcal{D}) \) from \((\Theta(C), \Theta(f))\) to \((\Theta(C'), \Theta(f'))\) given by the diagrams

\[
\begin{array}{cc}
\Theta(C) & \xrightarrow{\Theta(f(g))} g\Theta(C) \\
\alpha \downarrow & \downarrow g\alpha \\
\Theta(C') & \xrightarrow{\Theta(f'(g))} g\Theta(C')
\end{array}
\]

Since \( \Theta \) is full, there exists a map \( C \xrightarrow{\alpha'} C' \) such that \( \Theta(\alpha') = \alpha \). Thus there is a map in \( \text{Cat}(\tilde{G}, \mathcal{C}) \)

\[
\begin{array}{cc}
C & \xrightarrow{f(g)} gC \\
\alpha' \downarrow & \downarrow g\alpha' \\
C' & \xrightarrow{f'(g)} gC'
\end{array}
\]

whose image is the map above. This gives fullness. \( \square \)

The proposition shows that the homotopy fixed point construction for a \( G \)-category \( \mathcal{C} \) is homotopy invariant.
2.4 Pseudo equivariance

2.4.1 Pseudo equivariant functors

Let $\mathcal{C}$ and $\mathcal{D}$ be $G$-categories. We define the notion of a pseudo equivariant functor $\Theta: \mathcal{C} \to \mathcal{D}$, and we then show that such a functor induces an on the nose equivariant functor

$$\text{Cat}(\tilde{G}, \mathcal{C}) \longrightarrow \text{Cat}(\tilde{G}, \mathcal{D}).$$

Thus it induces maps on fixed points

$$\text{Cat}(\tilde{G}, \mathcal{C})^H \longrightarrow \text{Cat}(\tilde{G}, \mathcal{D})^H$$

for all subgroups $H \subseteq G$.

**Definition 2.4.1.** A pseudo equivariant functor between $G$-categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $\Theta: \mathcal{C} \to \mathcal{D}$, together with a natural isomorphism of functors $\theta_g: \Theta \circ g \cong g \circ \Theta$ for every $g \in G$, with $\theta_e = \text{id}$, such that the following diagram commutes for all $g, h \in G$:

$$\begin{align*}
\Theta(ghC) & \xrightarrow{\theta_{gh}(C)} g\Theta(hC) \xrightarrow{g\theta_h(C)} gh\Theta(C) \\
\Theta(g(hC)) & \xrightarrow{\theta_g(hC)} g\Theta(hC) \xrightarrow{g\theta_h(C)} gh\Theta(C)
\end{align*}$$

**Remark 2.4.2.** If $\theta_g$ are equalities for all $g \in G$, then $\Theta$ is actually an equivariant functor.

**Remark 2.4.3.** We explain the choice of nomenclature. Recall that a $G$-category is a functor $G \to \text{Cat}$, and an equivariant map between $G$-categories is then just a natural transformations of such functors. A pseudo equivariant map between $G$-categories is just a pseudo natural transformation between the two functors $G \to \text{Cat}$.

We claim that a pseudo equivariant functor $\Theta: \mathcal{C} \to \mathcal{D}$ induces an equivariant map after
applying the Cat($\tilde{G}, -$) functor.

**Proposition 2.4.4.** A pseudo equivariant functor $\Theta: C \to D$, naturally induces an equivariant functor

\[ \tilde{\Theta}: \text{Cat}(\tilde{G}, C) \longrightarrow \text{Cat}(\tilde{G}, D). \]

**Proof.** Clearly post composing a functor $F: \tilde{G} \longrightarrow C$ with $\Theta$ does not yield an equivariant functor, but we can use the natural isomorphisms $\theta_g$ to create one. Define

\[ \tilde{\Theta}(F)(g) = g\Theta(g^{-1}F(g)) \]

Recall that there is a unique map in $\tilde{G}$ from $g$ to $g'$, which we denote by $(g', g)$. Applying $\Theta \circ F$ we get a map $\Theta(F(g)) \xrightarrow{\Theta(F(g'), g)} \Theta(F(g'))$ in $D$. We define $\tilde{\Theta}(g', g)$ to be the composite

\[ g\Theta(g^{-1}F(g)) \xrightarrow{\theta_g^{-1}} \Theta(gg^{-1}F(g)) \xrightarrow{\Theta(F(g'), g)} \Theta(g'g^{-1}F(g')) \xrightarrow{\theta_{g'}} g'\Theta(g'^{-1}F(g')). \]

It is not hard to check that with these definitions $\tilde{\Theta}$ is an equivariant functor. \qed

**Corollary 2.4.5.** A pseudo equivariant functor $\Theta: C \longrightarrow D$, induces functors $C^{hH} \to D^{hH}$ on homotopy fixed points for all $H \subseteq G$.

**Question 2.4.6.** Does every equivariant functor $\text{Cat}(\tilde{G}, C) \to \text{Cat}(\tilde{G}, D)$ come from a pseudo equivariant functor $C \to D$?

### 2.4.2 Induced map on homotopy fixed points

We write down the explicit map $\tilde{\Theta}: \text{Cat}(\tilde{G}, C) \to \text{Cat}(\tilde{G}, D)$ induced from a pseudo equivariant functor $\Theta: C \to D$ on fixed points, because it will be useful later on. The idea is that we will encounter interesting maps in $K$-theory, which turn out to be fixed point maps.
of equivariant $K$-theory maps that arise from pseudo equivariant functors on the categorical level.

Recall the explicit description of homotopy fixed points given in Theorem 2.3.10. Let $(C, f)$ be an object in $\text{Cat}(\tilde{G}, \mathcal{C})^H$. Under the induced map on $H$-fixed points

$$\tilde{\Theta}^H : \text{Cat}(\tilde{G}, \mathcal{C})^H \rightarrow \text{Cat}(\tilde{G}, \mathcal{D})^H,$$

this gets sent to $(\Theta(C), f_\theta)$ where $f_\theta(g)$ is defined as the composite

$$\Theta(C) \xrightarrow{\Theta(f(g))} \Theta(gC) \xrightarrow{\theta_g} g\Theta(C).$$

Since $f(e) = \text{id}$ and $\theta_e = \text{id}$, it follows immediately that $f_\theta(e) = \text{id}$. To show that $f_\theta$ satisfies the cocycle condition, we use the fact that $f$ satisfies it, together with the diagram in definition 2.4.1. By that diagram, the maps in the following composite themselves factor as composites:

$$
\begin{array}{c}
\Theta(C) \\
\downarrow \Theta(f(g)) \\
\Theta(gC) \\
\downarrow \Theta(gf(h)) \\
\Theta(ghC) \\
\downarrow \theta_{gh}(C) \\
g\Theta(C) \\
\end{array}
\begin{array}{c}
\Theta(f(gh)) \\
\Theta(gC) \\
\Theta(gf(h)) \\
\theta_g(hC) \\
g\Theta(hC) \\
\theta_g(hC) \\
g\Theta(C) \\
\end{array}
\begin{array}{c}
\Theta(ghC) \\
\theta_{gh}(C) \\
g\Theta(C) \\
\theta_g(hC) \\
g\Theta(hC) \\
\end{array}
\begin{array}{c}
g\Theta(C) \\
g\Theta(f(h)) \\
g\Theta(hC) \\
\end{array}
$$

We can use the naturality diagram for $\theta_g$

$$
\begin{array}{c}
\Theta(gC) \\
\downarrow \theta_g(C) \\
g\Theta(C) \\
\downarrow \theta_g(f(h)) \\
g\Theta(f(h)) \\
\end{array}
\begin{array}{c}
\Theta(ghC) \\
\theta_g(hC) \\
g\Theta(hC) \\
\end{array}
\begin{array}{c}
\Theta(ghC) \\
\theta_g(hC) \\
g\Theta(C) \\
\theta_g(hC) \\
g\Theta(hC) \\
\end{array}
$$

to replace the middle maps in the diagram above and we get that
\[ \Theta(C) \xrightarrow{\Theta(f(h))} \Theta(ghC) \xrightarrow{\theta_{gh}(C)} gh\Theta(C) \]

is the same as

\[ \Theta(C) \xrightarrow{\Theta(f(g))} \Theta(gC) \xrightarrow{\theta_{g}(C)} g\Theta(C) \xrightarrow{g\Theta(f(h))} g\Theta(hC) \xrightarrow{g\theta_{h}(C)} gh\Theta(C). \]

Thus \( f_\theta(gh) = (g \cdot f_\theta(h))f_\theta(g). \)

### 2.4.3 Another consequence of pseudo equivariance

We can use pseudo equivariance to weaken the hypothesis of Proposition 2.3.12 from requiring the functor to be on the nose equivariant to requiring it to be pseudo equivariant. Surprisingly, we get the same conclusion.

**Proposition 2.4.7.** Let \( \Theta : \mathcal{C} \to \mathcal{D} \) be a pseudo equivariant functor, which is a nonequivariant equivalence. Then the induced functor

\[ \text{Cat}(\tilde{G}, \mathcal{C}) \to \text{Cat}(\tilde{G}, \mathcal{D}) \]

is a weak \( G \)-weak equivalence.

**Proof.** The equivariant map

\[ \tilde{\Theta} : \text{Cat}(\tilde{G}, \mathcal{C}) \to \text{Cat}(\tilde{G}, \mathcal{D}), \]

given in Proposition 2.4.4 is a nonequivariant equivalence with inverse \( \tilde{\Theta}^{-1} \).
We have a commutative diagram:

\[
\begin{array}{ccc}
\text{Cat}(\tilde{G}, \mathcal{C}) & \xrightarrow{\tilde{\Theta}} & \text{Cat}(\tilde{G}, \mathcal{D}) \\
\iota \downarrow & & \downarrow \iota \\
\text{Cat}(\tilde{G}, \text{Cat}(\tilde{G}, \mathcal{C})) & \xrightarrow{\text{Cat}(\tilde{G}, -)(\tilde{\Theta})} & \text{Cat}(\tilde{G}, \text{Cat}(\tilde{G}, \mathcal{D}))
\end{array}
\]

By Proposition 2.3.12, the bottom map is a weak $G$-equivalence, and by Lemma 2.3.7 the vertical maps are $G$-equivalences. Therefore the top map is a weak $G$-equivalence.

\[\square\]

**Corollary 2.4.8.** A pseudo equivariant functor $\Theta: \mathcal{C} \to \mathcal{D}$ which is a nonequivariant equivalence induces equivalences of homotopy fixed points

\[\mathcal{C}^hH \to \mathcal{D}^hH\]

for all $H \subseteq G$.

### 2.5 The homotopy fixed points of a group

The homotopy fixed point category $\mathcal{C}^hG$ simplifies enormously when $\mathcal{C} = \Pi$, a topological group regarded as a topological category with one object, with $G$-action. In that case, the homotopy fixed points can be interpreted in terms of the well-known notion of *crossed group homomorphisms*. The category $G\mathcal{C}at(\tilde{G}, \Pi)$ has been studied extensively in [GMM], and we review here the relevant interpretations that we have worked out there. These interpretations we obtain in this case will be necessary for later results.

**Definition 2.5.1.** Let $G$ and $\Pi$ be topological groups, so that $G$ acts on $\Pi$. A continuous map $\alpha: G \to \Pi$ is a *crossed homomorphism* if

\[\alpha(gh) = \alpha(g)(g \cdot \alpha(h))\] (2.7)
for all $g, h \in G$.

Observe that for a crossed homomorphism, we have

$$\alpha(e) = e, \quad \alpha(g)^{-1} = g \cdot \alpha(g)^{-1} \quad \text{and} \quad \alpha(g^{-1})^{-1} = g^{-1} \cdot \alpha(g).$$ (2.8)

Note from the description of objects of the homotopy fixed point category given in Theorem 2.3.10 that the object space of $\Pi^{hG} = \text{Cat}(\tilde{G}, \Pi)^G$ is precisely the space of crossed homomorphism $G \to \Pi$.

**Definition 2.5.2.** Let $G$ act on $\Pi$. Define the crossed functor category $\mathcal{C}at_x(G, \Pi)$ to be the category whose objects are the crossed homomorphisms $G \to \Pi$ and whose morphisms $\sigma: \alpha \to \beta$ are the elements $\sigma \in \Pi$ such that

$$\beta(g)(g \cdot \sigma) = \sigma \alpha(g).$$

The composite $\tau \circ \sigma, \tau: \beta \to \gamma$ is given by $\tau \sigma$.

Note that every morphism in $\mathcal{C}at_x(\tilde{G}, \Pi)$ is an isomorphism. Remark 2.3.11 translates to the following statement in this case.

**Remark 2.5.3.** If the action of $G$ on $\Pi$ is trivial, then the crossed functor category is just the functor category $\mathcal{C}at(G, \Pi)$ since homomorphisms $\alpha: G \to \Pi$ correspond to functors $\alpha: G \to \Pi$ and elements $\sigma \in \Pi$ such that $\beta(g)\sigma = \sigma \alpha(g)$ for $g \in G$ correspond to natural transformations $\alpha \to \beta$.

We rephrase Theorem 2.3.10 when the category $\mathcal{C}$ is a group $\Pi$ with $G$-action in terms of Definition 2.5.1.

**Theorem 2.5.4.** The fixed point category $G\text{Cat}(\tilde{G}, \Pi) = \text{Cat}(\tilde{G}, \Pi)^G$ is isomorphic to the crossed functor category $\mathcal{C}at_x(G, \Pi)$.  

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Corollary 2.5.5. For $H \subset G$, the fixed point category $\mathcal{Cat}(\tilde{G}, \Pi)^H$ is equivalent to the crossed functor category $\mathcal{Cat}_x(H, \Pi)$.

2.5.1 When is $\iota: \mathcal{A} \to \mathcal{Cat}(\tilde{G}, \mathcal{A})$ a weak $G$-equivalence?

Recall the map

$$\iota: \mathcal{A} \simeq \mathcal{Cat}(\ast, \mathcal{A}) \to \mathcal{Cat}(\tilde{G}, \mathcal{A})$$

from (2.3). It is a nonequivariant equivalence defined for any $G$-category $\mathcal{A}$. We analyze this for $\mathcal{A} = \Pi$ a group with $G$-action, and give conditions for when it is an equivalence.

The material from this section is also from [GMM].

Since $\mathcal{Cat}(\tilde{G}, \Pi)^G$ is a groupoid, it is equivalent to the coproduct of its subcategories $\text{Aut}(\alpha)$, where we choose one $\alpha$ from each isomorphism class of objects. The following definition is standard.

Definition 2.5.6. The first non-abelian cohomology group $H^1(G; \Pi_G)$ is the pointed set of isomorphism classes of crossed homomorphisms $G \to \Pi$. We write $[\alpha]$ for the isomorphism class of $\alpha$. The basepoint of $H^1(G; \Pi_G)$ is $[\varepsilon]$, where $\varepsilon$ is the trivial crossed homomorphism given by $\varepsilon(g) = e$ for $g \in G$.

With this language, Corollary 2.5.5 can be restated as follows.

Theorem 2.5.7. For $H \subset G$, $\mathcal{Cat}(\tilde{G}, \Pi)^H$ is equivalent to the coproduct of the categories $\text{Aut}(\alpha)$, where the coproduct runs over $[\alpha] \in H^1(H; \Pi_H)$.

Here $\text{Aut}(\alpha)$ implicitly refers to the ambient group $\Pi \rtimes H$, not $\Gamma = \Pi \rtimes G$. By (2.4), we obtain the same group $\text{Aut}(\alpha)$ for $\alpha$ considered as an object of $\mathcal{Cat}(\tilde{K}, \Pi)^H$ for any $H \subset K \subset G$.

Define the centralizer $\Pi^\alpha$ of a crossed homomorphism $\alpha: G \to \Pi$ to be the subgroup

$$\Pi^\alpha = \{ \sigma \in \Pi | \alpha(g)(g \cdot \sigma) = \sigma \alpha(g) \text{ for all } g \in G \}$$
of $\Pi$. It is the automorphism group $\text{Aut}(\alpha)$ of the object $\alpha$ in $\mathcal{C}at_{\mathcal{X}}(G, \Pi)$.

If the action is trivial,

$$\Pi^\alpha = \{ \sigma \in \Pi | \sigma^{-1} \alpha(g) \sigma = \alpha(g) \text{ for all } g \in G \}$$

is the usual centralizer of $\alpha$ in $\Pi$.

Note that when $\mathcal{A} = \Pi$ for a $G$-group $\Pi$, $\iota$ sends the unique object of $\Pi$ to the basepoint $[\varepsilon] \in H^1(G; \Pi)$. As a special case, $\text{Aut}(\varepsilon) = \Pi^G$ and $\iota^G$ restricts to the identity functor from $\Pi^G$ to $\text{Aut}(\varepsilon)$. This implies the following result.

**Proposition 2.5.8.** The functor $\iota^G : \Pi^G \to \mathcal{C}at(\tilde{G}, \Pi)^G$ is an equivalence of categories if and only if $H^1(G; \Pi_G) = [\varepsilon]$.

We will see in section 4.5 that in the case of a Galois extension $E$ with Galois group $G$ this says that the algebraic $K$-theory fixed point spectrum $\mathbb{K}_G(E)^H$ is equivalent to $\mathbb{K}(E^H)$, since Hilbert’s theorem 90 gives that that $H^1(G, GL_n(E)_G)$ is trivial. We will include a proof of Hilbert’s theorem 90 using faithfully flat descent in Section 4.5.4.

### 2.6 From categorical to space-level homotopy fixed points

In this section we give conditions for when the homotopy fixed points commute with the classifying space functor, namely we ask, when is it true that

$$BCat(\tilde{G}, \mathcal{C}) \simeq \text{Map}(EG, B\mathcal{C})?$$

We emphasize that this is not always the case. The results in this section are restated from [GMM].

There is an obvious comparison map relating the categorical and space level constructions.
For any $G$-categories $\mathcal{A}$ and $\mathcal{C}$, we have the evaluation $G$-functor
\[ \varepsilon: \text{Cat}(\mathcal{A}, \mathcal{C}) \times \mathcal{A} \to \mathcal{C}. \]

Applying the classifying space functor and taking adjoints, this gives a $G$-map
\[ \xi: B\text{Cat}(\mathcal{A}, \mathcal{C}) \to \text{Map}(B\mathcal{A}, B\mathcal{C}). \] \( (2.9) \)

When $\mathcal{A}$ and $\mathcal{C}$ are both discrete (in the topological sense), there is a simple analysis of this map in terms of the simplicial mapping space $\text{sSet}(N\mathcal{A}, N\mathcal{C})$. The following two lemmas are well-known nonequivariantly.

**Lemma 2.6.1.** For categories $\mathcal{A}$ and $\mathcal{C}$, there is a natural isomorphism
\[ \mu: N\text{Cat}(\mathcal{A}, \mathcal{C}) \cong \text{sSet}(N\mathcal{A}, N\mathcal{C}), \]
and this is an isomorphism of simplicial $G$-sets if $\mathcal{A}$ and $\mathcal{C}$ are $G$-categories.

**Proof.** Let $\Delta_n$ be the poset $\{0, 1, \cdots, n\}$, viewed as a category. The $n$-simplices of $\text{Cat}(\mathcal{A}, \mathcal{C})$ are the functors $\Delta_n \to \text{Cat}(\mathcal{A}, \mathcal{C})$. By adjunction, they are the functors $\mathcal{A} \times \Delta_n \to \mathcal{C}$. Since $N$ is full and faithful, these functors are the maps of simplicial sets
\[ N\mathcal{A} \times N\Delta_n \cong N(\mathcal{A} \times \Delta_n) \to N\mathcal{C}. \]

By definition, these maps are the $n$-simplices of $\text{sSet}(N\mathcal{A}, N\mathcal{C})$. These identifications give the claimed isomorphism of simplicial sets. The compatibility with the actions of $G$ when $\mathcal{A}$ and $\mathcal{C}$ are $G$-categories is clear. \qed
Lemma 2.6.2. For simplicial sets $K$ and $L$, there is a natural map

$$\nu: |\text{sSet}(K, L)| \to \text{Map}(|K|, |L|).$$

If $K$ and $L$ are simplicial $G$-sets, $\nu$ is a map of $G$-spaces, and it is a weak $G$-equivalence when $L$ is a Kan complex.

Proof. The evaluation map $\text{sSet}(K, L) \times K \to L$ induces a map

$$|\text{sSet}(K, L)| \times |K| \cong |\text{sSet}(K, L) \times K| \to |L|$$

whose adjoint is $\nu$. When $L$ is a Kan complex, so is $\text{sSet}(K, L)$ (e.g. [May92, 6.9]), and the natural maps $L \to \text{Sing} |L|$ and $\text{sSet}(K, L) \to \text{Sing} |\text{sSet}(K, L)|$ are homotopy equivalences, where $\text{Sing}$ is the total singular complex functor. A diagram chase shows that $\nu$ induces a bijection on homotopy classes of maps

$$\xi_*: [|J|, |\text{sSet}(K, L)|] \to [|J|, \text{Map}(|K|, |L|)]$$

for any simplicial set $J$. Letting $G$ act trivially on $J$, all functors in sight commute with passage to $H$-fixed points, and the equivariant conclusions follow. \hfill \square

Combining Lemma 2.6.1 and Lemma 2.6.2, and noting that if $\mathcal{C}$ is a groupoid, then its nerve is a Kan complex, we get the following result.

Proposition 2.6.3. For $G$-categories $\mathscr{A}$ and $\mathcal{C}$, the map $\xi$ of (2.9) is a weak $G$-equivalence if $\mathcal{C}$ is a groupoid.

Example 2.6.4. If $\mathcal{C} = \Pi$, a discrete group with $G$-action, Proposition 2.6.3 gives a weak $G$-equivalence

$$\text{BCat}(\tilde{G}, \Pi) \simeq \text{Map}(EG, B\Pi).$$
The following proposition is an immediate consequence of Proposition 2.6.3.

**Proposition 2.6.5.** If \( \mathcal{C} \) is a \( G \)-groupoid, then the homotopy fixed points commute with the classifying space functor, namely,

\[
B(\mathcal{C}^h_H) \simeq (B\mathcal{C})^h_H
\]

for any \( H \subseteq G \).

### 2.7 Homotopy fixed points of module categories

#### 2.7.1 \( G \)-rings and twisted group rings

A \( G \)-ring is a ring \( R \) with a left action of \( G \) by ring automorphisms. If \( R \) is a topological ring, we ask for the action to be through continuous ring automorphisms. We have a homomorphism \( G \to \text{Aut}(R) \), and we write \( g(r) = r^g \) for the automorphism \( g: R \to R \) determined by \( g \in G \). Then \( r^{gh} = g(h(r)) = (r^h)^g \).

**Example 2.7.1.** When \( R \) is a subquotient of \( \mathbb{Q} \), the only automorphism of \( R \) is the identity and the action of \( G \) must be trivial.

However, we will see that even trivial \( G \)-actions on rings will yield nontrivial equivariant algebraic \( K \)-theory. For example, the topological rings \( \mathbb{R} \) and \( \mathbb{C} \) with trivial actions will yield equivariant topological real and complex \( K \)-theory. The equivariant algebraic \( K \)-theory of discrete rings studied in [FHM82] and [DK82] back in the 1980’s was all for rings with trivial \( G \)-action.

**Example 2.7.2.** An important nontrivial example is the action of the Galois group on a Galois extension \( E \) of a field \( F \), or on the number ring \( \mathcal{O}_E \) of \( E \).

**Example 2.7.3.** Another nontrivial example is the topological ring \( \mathbb{C} \) with \( \mathbb{Z}/2\mathbb{Z} \) conjugation action.
Suppose that $R$ is a commutative $G$-ring with action given by $\theta: G \to \text{Aut}(R)$. Observe that $R$ is an $R^G$-algebra, where $R^G$ is the subring of $G$-invariants. We can reinterpret $\theta$ as a group homomorphism $\theta: G \to \text{End}_{R^G} R$, and ask the question of when we can extend this to a ring map. More precisely, we seek to put a ring structure on the underlying abelian group of the group ring $R[G]$, for which the map $\theta$ extends to a ring map.

This naturally leads to the definition of twisted group ring (or skew group ring), which we will denote by $R^G[G]$ (it is variously denoted in the literature also as $R \rtimes G$ or $R \ast G$). A more precise notation that takes into the action of $G$ on $R$ given by the homomorphism $\theta: G \to \text{Aut}(R)$ would be $R_{\theta}[G]$. However, the action of $G$ on $R$ will many times be implicit, so we will not adopt this more pendantic notation.

**Definition 2.7.4.** As an $R$-module, the twisted, or skew, group ring $R^G[G]$ is the same as the group ring $R[G]$, which is the case when $G$ acts trivially on $R$. We define the product on $R^G[G]$ by $R^G$-linear (not $R$-linear) extension of the relation

$$(rg)(sh) = rs^g gh$$

for $r, s \in R$ and $g, h \in G$.

Thus moving $g$ past $s$, “twists” the ring element by the group action. Note that $R$ and $R^G[G]$ are subrings of $R^G[G]$ and

$$gr = r^g g.$$ 

Observe that the definition of the twisted multiplication in $R^G[G]$ is precisely what enables us to extend the group homomorphism $\theta: G \to \text{End}_{R^G} R$ to a ring homomorphism

$$\theta: R^G[G] \to \text{End}_{R^G} R, \quad (r g) \mapsto (s \mapsto r s^g).$$
2.7.2 Modules over twisted group rings

**Definition 2.7.5.** We call (left) \( R_G[G] \)-modules *G-ring modules* or *skew G-modules*.

Note that \((rs)^g = (r^g)(s^g)\) for all \( r, s \in R \), thus \( R \) is an example of an \( R_G[G] \)-module.

The following observation is immediate from the definition of the twisted group ring.

**Observation 2.7.6.** An \( R_G[G] \)-module \( M \) is a left \( R \)-module with a *semilinear* \( G \)-action, i.e., \( g(rm) = r^g(gm) \) for \( m \in M \). If the action of \( G \) on \( R \) is trivial, then an \( R[G] \)-module is a left \( R \)-module \( M \) with *linear* \( G \)-action, namely, \( g(rm) = r(gm) \).

From this point of view an \( R_G[G] \)-linear map of \( R_G[G] \)-modules \( f : M \to N \) is a map of \( R \)-modules, which commutes with the \( G \)-action.

If \( G \) is finite and \(|G|\) is invertible in \( R \), we obtain the following characterization of projective modules over \( R_G[G] \), which will be crucial in our applications to \( K \)-theory of \( G \)-rings.

**Proposition 2.7.7.** If \( G \) is finite and \(|G|^{-1} \in R \), then an \( R_G[G] \)-module is projective if and only if it is projective as an \( R \)-module.

**Proof.** An \( R_G[G] \)-module \( M \) is projective if and only if the functor

\[
\text{Hom}_{R_G[G]}(M, -) : \text{Mod}(R_G[G]) \to \text{Mod}(R_G[G])
\]

is exact. This functor is always left exact, and it is also right exact precisely when \( M \) is projective. Let \( M \) and \( N \) be \( R_G[G] \)-modules. As noted in Observation 2.7.6, \( M \) and \( N \) are \( R \)-modules with semilinear \( G \)-action. Then the \( R_G[G] \)-module \( \text{Hom}_{R_G[G]}(M, N) \) is the \( R \)-module \( \text{Hom}_R(M, N) \) with semilinear \( G \)-action given by conjugation, i.e., for an \( R \)-linear map \( f : M \to N \), \( g f(m) = g(f(g^{-1}m)) \). Again from Observation 2.7.6, we have that

\[
\text{Hom}_{R_G[G]}(M, N) \cong \text{Hom}_R(M, N)^G.
\]

The fixed point functor \((-)^G\) on \( R_G[G] \)-modules is right exact when the order of \( G \) is
invertible in $R$. Thus when $|G| \in R$, the functor $\text{Hom}_{R[G]}(M, -)$ is exact precisely when the functor $\text{Hom}_R(M, -)$ is exact.

Of course, we do not have a similar statement for free modules. Clearly, a free $R[G]$-module is free over $R$, but the converse is not true: Freeness over $R$ definitely does not imply freeness over $R[G]$. For a set $A$, let $R[A]$ denote the free $R$-module on the basis $A$. The following proposition shows how we can put an $R[G]$-module structure on $R[A]$ if $A$ is a $G$-set; this is equivalent to specifying a semilinear $G$-action on $R[A]$.

**Proposition 2.7.8.** Let $A$ be a $G$-set and define

$$g(\sum_a r_a a) = \sum_a r^a g a$$

for $g \in G$, $r_a \in R$, and $a \in A$. Then $R[A]$ is an $R[G]$-module.

We give a classification of $R[G]$-module structures on free rank $n$ $R$-modules in terms of the homotopy fixed point category of the group $GL_n(R)$, regarded as a single object groupoid. It inherits a $G$-action from the $G$-action on $R$.

**Theorem 2.7.9.** Let $R$ be a $G$-ring. Then the set of isomorphism classes of $R[G]$-module structures on the $R$-module $R^n$ is in canonical bijective correspondence with the isomorphism classes of objects in the homotopy fixed point category $GL_n(R)^{hG} = \text{Cat}(\tilde{G}, GL_n(R))^G$.

**Proof.** From Corollary 2.5 we have that $\text{Cat}(\tilde{G}, GL_n(R))^G \cong \mathcal{C}at_x(G, GL_n(R))$, the category of crossed homomorphisms $\theta : G \to GL_n(R)$. Two crossed homomorphisms $\theta_1$ and $\theta_2$ are isomorphic if there is an element $\pi \in \Pi$ such that

$$(\pi^{-1})\theta_1(g)(g \cdot \pi) = \theta_2(g)$$

for all $g \in G$. 

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Let \( \{e_i\} \) be the standard basis for \( \mathbb{R}^n \). We show that the formula
\[
ge_i = \rho(g)(e_i)
\]
establishes a bijection between \( R_G[G] \)-module structures on \( \mathbb{R}^n \) and crossed homomorphisms \( \rho: G \to GL(n, \mathbb{R}) \). Moreover, two \( R_G[G] \)-modules with underlying \( \mathbb{R} \)-module \( \mathbb{R}^n \) are isomorphic if and only if their corresponding crossed homomorphisms are isomorphic.

Given an \( R_G[G] \)-module structure on \( \mathbb{R}^n \), define the matrix \( \rho(g) \) in \( GL(n, \mathbb{R}) \) by letting its \( i^{th} \) column be \( (s_{i,j}) \), where
\[
ge_i = \sum_j s_{i,j}e_j.
\]
Conversely, given \( \rho \), write \( \rho(g) = (s_{i,j}) \) and define \( ge_i \) by the same formula. From either starting point, we have \( ge_i = \rho(g)(e_i) \). For a second element \( h \in G \), write \( \rho(h) = (t_{i,j}) \), where \( \rho(h) \) is either determined by an \( R_G[G] \)-module structure or is given by a crossed homomorphism \( \rho \). Since \( gr = r^g g \) in \( R_G[G] \) and \( g(r_{i,j}) = (r^g_{i,j}) \) in \( GL(n, \mathbb{R}) \), the relation \( (gh)e_i = g(he_i) \) required of an \( R_G[G] \)-module is the same as the relation
\[
\rho(g)\rho(h)(e_i) = \rho(g)(\rho(h))(e_i)
\]
required of a crossed homomorphism. Indeed, \( (gh)e_i = \rho(gh)(e_i) \) and
\[
(gh)e_i = g(he_i) = \sum_j g(t_{i,j}e_j)
= \sum_j t^g_{i,j}ge_j = \sum_j t^g_{i,j}s_{j,k}e_k
= \rho(g)(\sum_j t^g_{i,j}e_j) = \rho(g)(\rho(h)(e_i)).
\]
The remaining compatibilities, in particular for the transitivity relation required of a mod-
ule, are equally straightforward verifications, as is the verification of the statement about isomorphisms.

In view of Proposition 2.7.8, this has the following immediate consequence.

**Corollary 2.7.10.** For a $G$-ring $R$, any $n$-pointed $G$-set $A$ canonically gives rise to a crossed homomorphism $\rho_A : G \to GL(n, R)$.

### 2.7.3 The category $\text{Cat}(\tilde{G}, \text{Mod}(R))$

For a $G$-ring $R$, the category of finitely generated $R$-modules $\text{Mod}(R)$ becomes a $G$-category with action defined in the following way. Let $M$ be an $R$-module with action map $\gamma : R \times M \to M$. Then we let $gM = M$ as abelian groups, and we define the action map by pulling back the action on $M$ along $g : R \to R$:

$$
\gamma_g : R \times M \xrightarrow{g \times \text{id}} R \times M \xrightarrow{\gamma} M.
$$

This twists the $R$-action on $M$ by the action of $G$ on $R$. Explicitly, the $R$ action on $gM$, which we will denote by $\cdot_g$ to differentiate from the $R$-action on $M$, is given by

$$
r \cdot_g m := r^g m,
$$

where on the right hand side of the equation we are using the action of $R$ on $M$.

We note that

$$
RG[G] \otimes_R M \cong \bigoplus_{g \in G} gM.
$$

For a morphism $f : M \to N$, we define $gf : gM \to gN$ by $(gf)(m) = f(m)$. Thus $gf$ is the same as $f$ as a homomorphism of abelian groups, but it interacts differently from $f$ with the scalar multiplication.
Note that in general $M$ is not necessarily isomorphic to $gM$ as $R$-modules. The identity of abelian groups $M = gM$ is not an $R$-linear map, since the $R$-action is different on the two sides of the equality. However, we do have an isomorphism of free $R$-modules $R^n \cong gR^n$, which plays an important role.

**Lemma 2.7.11.** The $R$-modules $R^n$ and $gR^n$ are isomorphic.

**Proof.** Let $\{e_i\}$ be the standard basis for $R^n$. Note that this is also a basis for $gR^n$: if $r_1^g e_1 + \cdots + r_n^g e_n = 0$, then $r_i^g = 0$ for all $i$, so $r_i = 0$ since $G$ acts by ring automorphisms. Also, every element in $gR^n$ can be written as

$$(r_1, \cdots, r_n) = (r_1^{(g^{-1})})^g e_1 + \cdots + (r_n^{(g^{-1})})^g e_n.$$ 

Now just define a map on basis elements as the identity $e_i \mapsto e_i$ and extend linearly, i.e. $re_i \mapsto r^g e_i$. \hfill \Box

We emphasize that the objects of the category $\text{Mod}(R)$ are $R$-modules $M$, which know nothing about the $G$-action on $R$. We used this action to define a $G$-action on the category $\text{Mod}(R)$, and now we will show how the $G$-category $\text{Mod}(R)$ relates to the category of modules over the twisted group ring $R_G[G]$, which by Observation 2.7.6 is the same as the category of $R$-modules with semilinear $G$-action.

**Proposition 2.7.12.** The homotopy fixed point category $\text{Mod}(R)^{hG}$ is equivalent to the category $\text{Mod}(R_G[G])$.

**Proof.** From the description of homotopy fixed point categories given in Theorem 2.3.9, the objects of the homotopy fixed point category $\text{Cat}(\tilde{G}, \text{Mod}(R))^G$ are $R$-modules $M$ together with compatible isomorphisms $f(g): gM \xrightarrow{\cong} M$, one for each element $g \in G$, for which
\( f(e) = \text{id}_M \) and which make the diagrams

\[
\begin{array}{ccc}
(gh)M & \xrightarrow{gf(h)} & gM \\
\downarrow f(gh) & & \downarrow f(g) \\
M & \rightarrow & f(g)
\end{array}
\]

commute.

Define an action of \( G \) on \( M \) by \( g \cdot m = f(g)(m) \). This is indeed an action since \( f(e) = \text{id}_M \) and

\[
(gh) \cdot m = f(gh)(m) \\
= f(g)gf(h)(m) \\
= f(g)f(h)(m) \\
= g \cdot (h \cdot m).
\]

The second to last identification is just the definition of the \( G \)-action on morphisms of modules in \( \text{Mod}(R) \); the morphism \( gf(h) \) is the same as \( f(h) \) as a morphism of abelian groups.

Now note that this action is indeed semilinear:

\[
g \cdot (rm) = f(g)(r \cdot g m) = r^g f(g)(m) = r^g (g \cdot m).
\]

Via this identification, the morphisms in the homotopy fixed point category are precisely the \( G \)-equivariant maps of \( G \)-modules.

Thus we have shown that the homotopy fixed point category \( \text{Cat}(\tilde{G}, \text{Mod}(R))^G \) can be identified with the category of modules with semilinear \( G \)-action. Combining this with Observation 2.7.6, we obtain the desired result. \( \square \)

Recall that, as explained in section 2.3.4, we have \( \text{Cat}(\tilde{G}, \text{Mod}(R))^H \simeq \text{Cat}(\tilde{H}, \text{Mod}(R))^H \),
so we immediately get the following corollary.

**Corollary 2.7.13.** The homotopy fixed point category $\text{Mod}(R)^{hH}$ is equivalent to the category $\text{Mod}(R_H[H])$ for all subgroups $H \subseteq G$.

Therefore, the $G$-category $\text{Cat}(\tilde{G}, \text{Mod}(R))$ encodes the module categories over the twisted group rings for all subgroups $H$ as fixed points. Thus by studying the equivariant object $\text{Cat}(\tilde{G}, \text{Mod}(R))$ we are implicitly studying the representation theory of the twisted group rings $R_H[H]$.

Let $\mathcal{P}(R)$ be the category of finitely generated projective $R$-modules. This becomes a $G$-category in the same way that $\text{Mod}(R)$ does since $gP$ is projective if $P$ is so: if $P \oplus Q \cong R^n$, then $gP \oplus gQ \cong gR^n \cong R^n$. The proof of Proposition 2.7.12 goes through to show that the category $\text{Cat}(\tilde{G}, \mathcal{P}(R))$ is equivalent to the category of finitely generated projective $R$-modules with semilinear $G$-action. Therefore, by Proposition 2.7.7, if $G$ is finite and the order of $G$ is invertible in $R$, we obtain Proposition 2.7.12 and its corollary if we restrict to the category of finitely generated $R$-modules.

**Proposition 2.7.14.** Suppose $G$ is finite and $|G|^{-1} \in R$. The homotopy fixed point category $\mathcal{P}(R)^{hG}$ is equivalent to the category $\mathcal{P}(R_G[G])$.

**Corollary 2.7.15.** Suppose $G$ is finite and $|G|^{-1} \in R$. The homotopy fixed point category $\mathcal{P}(R)^{hH}$ is equivalent to the category $\mathcal{P}(R_H[H])$ for all subgroups $H \subseteq G$.

### 2.7.4 The equivariant skeleton of free modules

If $M \cong R^n$, then $gM \cong gR^n \cong R^n$, so the $G$-action on $\text{Mod}(R)$ restricts to an action on the category $\mathcal{F}(R)$ of finitely generated free $R$-modules.

**Definition 2.7.16.** Let $\mathcal{F}(R)$ be the category with objects the based free $R$-modules $R^n$.
and morphism spaces

\[ \text{Mor}_{\mathcal{L}(R)}(R^n, R^m) = \begin{cases} \emptyset & \text{if } n \neq m \\ GL_n(R) & \text{if } n = m. \end{cases} \]

This is the same as the disjoint union of the one object categories \( GL_n(R) \), i.e.,

\[ \mathcal{L}(R) = \coprod_{n \geq 0} GL_n(R), \]

and it is a skeleton of the category of \( \text{iso}\mathcal{F}(R) \) of finitely generated free \( R \)-modules and isomorphisms.

We note that in general, even if \( C \) is a \( G \)-category, we do not necessarily have an action on the skeleton \( \text{sk} \mathcal{C} \), since the \( G \)-action is not closed on this subcategory. However, if \( R \) is a \( G \)-ring, we have an obvious action on \( \mathcal{L}(R) \): it is trivial on objects and on morphisms \( g \) acts entrywise. Clearly, the inclusion of the skeleton

\[ i: \mathcal{L}(R) \to \text{iso}\mathcal{F}(R) \]

is not an equivariant map since the object \( R^n \) is fixed in \( \mathcal{L}(R) \) but not in \( \text{iso}\mathcal{F}(R) \). However, we can define an inverse to it which is equivariant. Fix isomorphisms \( \gamma_M: M \cong R^k \) for all finitely generated free modules \( M \), i.e., fix a basis \( \{ \gamma_M^{-1}(e_i) = m_i \} \) for all \( M \) such that \( \gamma_M = \gamma_gM \) as isomorphisms of abelian groups. In other words, we pick the same basis for \( M \) and \( gM \); recall that \( M \) and \( gM \) are equal as abelian groups. We define \( i^{-1} \) by \( M \mapsto R^k \) on objects. Given an isomorphism \( M \to N \) in \( \text{iso}\mathcal{F}(R) \), it maps to the composite

\[ R^k \xrightarrow{\gamma_M^{-1}} M \xrightarrow{f} N \xrightarrow{\gamma_N} R^k. \]

We show that the map \( i^{-1} \) is equivariant. Clearly, it commutes with the \( G \)-action on
objects, since the action is trivial in \( \mathcal{L}(R) \) and if \( M \) has dimension \( k \) so does \( gM \). Now let \( f: M \to N \) be an isomorphism in \( \text{iso} \mathcal{F}(R) \), and suppose that

\[
m_i \mapsto r_{i1} n_1 + \cdots + r_{ik} n_k.
\]

The morphism \( gM \xrightarrow{gf} gN \) maps to

\[
R^k \xrightarrow{\gamma^{-1}_M} gM \xrightarrow{gf} gN \xrightarrow{\gamma_N} R^k.
\]

On basis elements, this is

\[
e_i \mapsto r_{i1} \cdot g e_1 + \cdots + r_{ik} \cdot g e_k = r^g_{i1} e_1 + \cdots + r^g_{ik}.
\]

Therefore, we get entrywise action by \( g \) on the matrix representing \( f \). Therefore, the map

\[
i^{-1} : \text{iso} \mathcal{F}(R) \to \mathcal{L}(R)
\]

is \( G \)-equivariant. It is a nonequivariant equivalence, thus by Proposition 2.3.12 we obtain the following result.

**Proposition 2.7.17.** Suppose \( R \) is a \( G \)-ring. Then there is a weak \( G \)-equivalence

\[
\text{Cat}(\tilde{G}, \text{iso} \mathcal{F}(R)) \to \text{Cat}(\tilde{G}, \mathcal{L}(R)).
\]

This will be extremely useful because it will allow us to use the skeleton \( \mathcal{L}(R) \) in equivariant algebraic \( K \)-theory without losing information about the entire category of free modules with its induced action of \( G \).
2.7.5 Equivariant Morita theory

We give a definition of *equivariant Morita equivalence*; the philosophy is that we want this notion to capture Morita equivalences of twisted group rings.

**Definition 2.7.18.** Two $G$-rings $R$ and $S$ are *equivariantly Morita equivalent* if they are nonequivariantly Morita equivalent and the equivalence

$$\text{Mod}(R) \to \text{Mod}(S)$$

is pseudo equivariant.

In [Bil12], Biland gives a definition of equivariant Morita equivalence, and it is easy to see that his definition agrees with ours\(^2\). Biland shows that Definition 2.7.18 is equivalent to having a $G$-equivariant bimodule, which provides the equivariant Morita equivalence. For the definition of $G$-bimodule and the details of the equivalence of the two statements we refer the reader to Biland’s preprint [Bil12, Thm. A].

Note that a consequence of our definition of equivariant Morita equivalence and Proposition 2.4.7 is the following proposition.

**Proposition 2.7.19.** If two $G$-rings $R$ and $S$ are equivariantly Morita equivalent, then there is an equivariant weak equivalence

$$\text{Cat}(\tilde{G}, \text{Mod}(R)) \to \text{Cat}(\tilde{G}, \text{Mod}(S)).$$

Thus we have a $G$-map which induces an equivalence on all fixed points

$$\text{Cat}(\tilde{G}, \text{Mod}(R))^H \to \text{Cat}(\tilde{G}, \text{Mod}(S))^H.$$
As we have shown in Proposition 2.7.12 this ensures that the twisted group rings $R_H[H]$ and $S_H[H]$ are Morita equivalent in the classical sense for all $H \subseteq G$.

We end with a consequence of equivariant Morita equivalence, which will be relevant in algebraic $K$-theory. Recall that a nonequivariant Morita equivalence $\text{Mod}(R) \to \text{Mod}(S)$ restricts to an equivalence $\mathcal{P}(R) \to \mathcal{P}(S)$ on the categories of finitely generated projective modules (for example, see [Wei13, II, 2.7]).

**Lemma 2.7.20.** If $R$ and $S$ are equivariantly Morita equivalent, then there is a weak $G$-equivalence

$$\text{Cat}(\tilde{G}, \mathcal{P}(R)) \to \text{Cat}(\tilde{G}, \mathcal{P}(S))$$

which induces equivalences of the homotopy fixed point categories of finitely generated projective modules $\mathcal{P}(R)^hH \to \mathcal{P}(S)^hH$ for all $H \subseteq G$.

**Proof.** Since $R$ and $S$ are equivariantly Morita equivalent, by definition, we have a nonequivariant Morita equivalence $\text{Mod}(R) \to \text{Mod}(S)$, which is pseudo equivariant. This restricts to an equivalence $\mathcal{P}(R) \to \mathcal{P}(S)$, which is pseudo equivariant, and we get the result by applying Proposition 2.4.7. The second statement follows by passing to fixed points. \qed


CHAPTER 3
EQUIVARIANT BUNDLES AND THEIR CLASSIFYING SPACES

3.1 Motivation

The results described in this chapter are joint work with B. Guillou and P. May, and we will refer the reader to our paper for some of the details. In [GMM], we find models for universal equivariant bundles and their classifying spaces as classifying spaces of categories. The reason why it is important to have such models is two-fold: they are needed in equivariant infinite loop space theory and in algebraic $K$-theory. We address how bundle theory comes into the picture for each of these two topics.

3.1.1 Equivariant infinite loop space theory

Infinite loop spaces satisfy a recognition principle: they are algebras over $E_\infty$-operads in Top (see [May72]). Algebras over an $E_\infty$-operad in $\text{Cat}$ are categories whose classifying spaces are, after group completion, infinite loop spaces. The same story carries through equivariantly for a finite group $G$. Equivariant infinite loop spaces (or infinite loop $G$-spaces) are $G$-spaces which have deloopings with respect to all finite dimensional representations of $G$, so they are zeroth spaces of genuine $G$-spectra. Equivariant infinite loop spaces are recognized as algebras over equivariant $E_\infty$-operads in $G\text{Top}$ (see [LMS86]).

A new development in equivariant infinite loop space theory is defining an $E_\infty$-operad in $G\text{Cat}$ such that algebras over it are $G$-categories whose classifying spaces are, once group completed, infinite loop $G$-spaces (see [GMa]). For this it is crucial to have models for equivariant universal bundles as classifying spaces of categories, as we go on to explain.

Nonequivariantly, an $E_\infty$-operad $\mathcal{O}$ in Top has spaces $\mathcal{O}(j) \simeq E\Sigma_j$, namely, universal $\Sigma_j$-bundles. An $E_\infty$-operad $\mathcal{O}$ in $\text{Cat}$ is defined by the property that the space-level operad $B\mathcal{O}$
with spaces $B\mathcal{O}(j)$ is an $E_\infty$-operad in $\text{Top}$. By Observation 2.3.6 we have that $B\tilde{\Sigma}_j \simeq E\Sigma_j$, thus the categorical operad $\mathcal{O}$ with categories $\mathcal{O}(j) = \tilde{\Sigma}_j$ is an $E_\infty$-operad. This is also known as the Barratt-Eccles operad, and algebras over $\mathcal{O}$ are permutative categories [May74].

The definition of an equivariant $E_\infty$-operad $\mathcal{O}_G$ in $G\text{Top}$ is in terms of equivariant universal bundles: the spaces $\mathcal{O}_G(j)$ are defined to be universal $(G, \Sigma_j)$-bundles, which we denote for now as $E(G, \Sigma_j)$. These are universal principal $\Sigma_j$-bundles, with total and base $G$-spaces, $G$-equivariant projection map, and commuting actions of $G$ and $\Sigma_j$ on the total space. We will give the precise definition in section 3.2. Models for universal equivariant bundles and their classifying spaces are described in [May96, VII], for example, but they are not given in terms of classifying spaces of categories.

An $E_\infty$-operad $\mathcal{O}_G$ in $G\text{Cat}$ is defined by the property that applying the classifying space functor levelwise yields an $E_\infty$-operad in $G\text{Top}$. Thus finding an $E_\infty$-operad in $G\text{Cat}$ amounts to finding models for equivariant universal principal $(G, \Sigma_j)$-bundles as classifying spaces of $G$-categories. We summarize this in Table 3.1 below.

<table>
<thead>
<tr>
<th></th>
<th>A nonequivariant $E_\infty$-operad $\mathcal{O}$</th>
<th>An equivariant $E_\infty$-operad $\mathcal{O}_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>in $\text{Top}$</td>
<td>has spaces universal $\Sigma_j$-bundles, i.e., $\mathcal{O}(j) \simeq E\Sigma_j$</td>
<td>has spaces universal $(G, \Sigma_j)$-bundles, i.e., $\mathcal{O}_G(j) \simeq E(G, \Sigma_j)$</td>
</tr>
<tr>
<td>example: $\mathcal{O}(j) = B\tilde{\Sigma}_j$</td>
<td>$B\mathcal{O}(j) \simeq E\Sigma_j$</td>
<td></td>
</tr>
<tr>
<td>in $\text{Cat}$</td>
<td>is defined such that $B\mathcal{O}(j) \simeq E\Sigma_j$</td>
<td>is defined such that $B\mathcal{O}_G(j) \simeq E(G, \Sigma_j)$</td>
</tr>
<tr>
<td>example: $\mathcal{O}(j) = \tilde{\Sigma}_j$</td>
<td>$\mathcal{O}(j) = \tilde{\Sigma}_j$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: $E_\infty$ operads
3.1.2 **Equivariant algebraic K-theory**

Quillen’s first definition of higher algebraic $K$-groups was as the homotopy groups of a space $BGL(R)^+$, which turns out to be homotopy equivalent to the basepoint component of the group completion of the topological monoid $B(\coprod_n GL_n(R)) = \coprod_n BGL_n(R)$. Note that this is the topological monoid of classifying spaces of principal $GL_n(R)$-bundles under Whitney sum. Equivariantly, we are unconcerned with any variant of Quillen’s original plus construction, but we instead replace the classifying spaces of principal $GL_n(R)$-bundles by classifying spaces of equivariant principal bundles, before group completion.

Note that in contrast to the equivariant bundles considered in the previous section, when $G$ was not acting on $\Sigma_j$ and we had commuting actions on the total space, now we are assuming that $G$ acts on $R$, which induces an action on $GL(R)$. The whole point is to take this action into account. The bundles which we are trying to understand are $(G, GL_n(R)_G)$-bundles; they are universal principal $GL_n(R)$ bundles, but they have twisted actions on the total space, i.e., they have an action of the semidirect product $GL_n(R) \rtimes G$ on the total space. The base space is a $G$-space and the projection map is $G$-equivariant (see Definitions 3.2.1 and 3.2.3).

The intuition of defining the equivariant algebraic $K$-theory space of a $G$-ring in terms of classifying spaces of $(G, GL_n(R)_G)$-bundles is right, in the sense that we are rigging the spaces to provide an algebra over an $E_\infty$-operad in $G\text{Top}$ that can be fed into an equivariant infinite loop space machine. This will be explained in section 4.4.

### 3.2 Equivariant bundles

Let $\Pi$ and $G$ be topological groups and suppose that we have an extension of groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \overset{q}{\longrightarrow} G \longrightarrow 1.$$  \hspace{1cm} (3.1)
There is a general theory of equivariant bundles corresponding such extensions (see, for example, [LM86, May90, May96]). However, we will only be interested in the case when $G$ acts on $\Pi$, the group $\Gamma$ is the semi-direct product $\Pi \rtimes G$, and the extension is split.

The underlying space of $\Gamma$ is $\Pi \times G$, and the product is given by

$$(\sigma, g)(\tau, h) = (\sigma(g \cdot \tau), gh).$$

We will refer to bundles corresponding to such extensions as $(G, \Pi_G)$-bundles: $G$ is the equivariance group, $\Pi$ is the structure group, and the subscript in $\Pi_G$ denotes that $G$ is acting on $\Pi$ and the bundle corresponds to the split extension given by the semidirect product with respect to this action. We give the precise definition of such bundles in the next section. If the action of $G$ on $\Pi$ is trivial, so that $\Gamma = \Pi \times G$, then we omit the subscript $G$, and refer to such bundles as $(G, \Pi)$-bundles.

Again, there is a general theory of $(G, \Pi_G)$-bundles [tD69, Las82, LM86, May96] corresponding to such extensions. The theory is especially familiar when $G$ acts trivially on $\Pi$. With $\Pi = O(n)$ or $U(n)$, the trivial action case gives classical equivariant bundle theory and equivariant topological $K$-theory.

### 3.2.1 Equivariant principal bundles

We shall only be interested in principal $(G, \Pi_G)$-bundles $p: E \to B$. A principal $(G, \Pi_G)$-bundle is nonequivariantly just a principal $\Pi$-bundle, but now there are $G$-actions in sight everywhere, including on the structure group $\Pi$, and they need to interact compatibly with the action of the structure group $\Pi$. We make this precise in the following definition.

1. In order to be consistent with [GMM], we do not use the notation from [May96] for bundles corresponding to extensions (3.1). Their notation is $(\Pi, \Gamma)$-bundles, namely the structure group is listed first and the extension group second. In the notation from [May96], the bundles we are considering are $(\Pi, \Pi \rtimes G)$-bundles.

2. The notation for bundles corresponding to extensions with $\Gamma = \Pi \times G$ is consistent with [May96], where they adopt the same convention for the trivial action case, and we felt that our notation for the general case better generalizes this.
Definition 3.2.1. Let $p: E \to B$ be a principal $\Pi$-bundle where $B$ is a $G$-space. Then $p$ is a principal $(G, \Pi_G)$-bundle if the (free) action of $\Pi$ on $E$ extends to an action of $\Gamma = \Pi \rtimes G$ and $p$ is a $\Gamma$-map, where $\Gamma$ acts on $B$ through the quotient map $\Gamma \to G$.

We say a few words about the $\Gamma$-action on the total space $E$. It is standard in equivariant bundle theory to let $G$ act from the left and $\Pi$ act from the right. Thus suppose that $X$ is a left $G$ and right $\Pi$ space (or category). Using elementwise notation, turn the right action of $\Pi$ into a left action by setting $\sigma x = x\sigma^{-1}$.

By an action of $\Gamma$ on $X$, we understand a left action that coincides with the given actions when restricted to the subgroups $G = e \times G$ and $\Pi = \Pi \times e$ of $\Gamma$.

The following condition gives a criterion for extending actions of $G$ and $\Pi$ on a space (or a category) $X$ to an action of the semi direct product $\Gamma = \Pi \rtimes G$.

Lemma 3.2.2. Given a right $\Pi$ and a left $G$ action on a space (or category) $X$, these extend to an action of $\Gamma = \Pi \rtimes G$ if and only if the given actions of $\Pi$ and $G$ satisfy the twisted commutation relation

$$g(x\sigma) = (gx)(g \cdot \sigma).$$

The placement of parentheses is crucial: we are taking group actions in different orders. When the action of $G$ on $\Pi$ is trivial, $g \cdot \sigma = \sigma$, this is the familiar statement that commuting left and right actions define an action by the product $\Pi \times G$.

Proof. Since $(\sigma, g) = (\sigma, e)(e, g)$, the action of $\Gamma$ must be defined by

$$(\sigma, g)x = (\sigma, e)(e, g)x = (\sigma, e)gx = \sigma gx = (gx)\sigma^{-1}.$$
the twisted commutation relation

\[(e, g)(\sigma, e) = (g \cdot \sigma, g) = (g \cdot \sigma, e)(e, g),\]

or the same relation with \(\sigma\) replaced by \(\sigma^{-1}\). Therefore (3.3) gives an action of \(\Gamma\) if and only if the given actions of \(\Pi\) and \(G\) satisfy the twisted commutation relation (3.2) \(\square\)

### 3.2.2 Equivariant universal bundles

**Definition 3.2.3.** A principal \((G, \Pi_G)\)-bundle \(p: E \to B\) is universal if for all \(G\)-spaces \(X\) of the homotopy types of \(G\)-CW complexes, pullback of \(p\) along \(G\)-maps \(f: X \to B\) induces a natural bijection from the set of homotopy classes of \(G\)-maps \(X \to B\) to the set of equivalence classes of \((G, \Pi_G)\)-bundles over \(X\).

There is a criterion that, modulo weak point-set topological conditions, describes universal \((G, \Pi_G)\)-bundles uniquely up to homotopy type. The following result is [LM86, Thm. 9], but the details of the proof are in [Las82, §2]. A principal \((G, \Pi_G)\)-bundle is numerable if it is trivial over the subspaces of \(B\) in a numerable open cover.

**Theorem 3.2.4.** A numerable principal \((G, \Pi_G)\)-bundle \(p: E \to B\) is universal if and only if \(E^\Lambda\) is contractible for all (closed) subgroups \(\Lambda\) of \(\Gamma\) such that \(\Lambda \cap \Gamma = \{e\}\).

Note that if \(G = e\), then the condition translates precisely into saying that the total space is contractible, which is the familiar nonequivariant condition for a principal bundle to be universal.

We refer the reader to [GMM] for a slightly more detailed commentary on the hypothesis of Theorem 3.2.4, and we content ourselves with saying that the numerability condition is satisfied if, for example, the base and total spaces are \(G\)-CW complexes.
3.3 Categorical models

3.3.1 Overview

Models for universal principal \((G, \Pi_G)\)-bundle are described in [May96, VII], for example, but our goal, as outlined in Section 3.1, is to find categories whose classifying spaces are the total and base spaces of a universal principal \((G, \Pi_G)\)-bundle.

Let \(G\) and \(\Pi\) be topological groups, and suppose \(G\) acts on \(\Pi\). This induces a left action on the category \(\tilde{\Pi}\); the action on objects forces the diagonal action on the morphism space \(\Pi \times \Pi\) in the same way as described in section 2.3.3.

Recall the functor \(\text{Cat}(\tilde{G}, -)\) from \(G\)-categories to \(G\)-categories defined in section 2.3.3. Define

\[
E(G, \Pi_G) = B\text{Cat}(\tilde{G}, \tilde{\Pi}).
\]

In section 3.3.2 we show that this is a \(\Pi \rtimes G\)-space. For a \(\Pi \rtimes G\)-category or \(\Pi \rtimes G\)-space, passage to orbits with respect to \(\Pi\) gives a \(G\)-category or a \(G\)-space. We define

\[
B(G, \Pi_G) = E(G, \Pi_G)/\Pi,
\]

and let

\[
p: E(G, \Pi_G) \to B(G, \Pi_G)
\]

be the projection map given by quotienting out by the \(\Pi\)-action.

We first note that, forgetting the \(G\)-action, we have a nonequivariant equivalence \(\text{Cat}(\tilde{G}, \tilde{\Pi}) \simeq \tilde{\Pi}\). However, this is still a \(\Pi\)-equivalence since \(\Pi\) acts freely on \(\tilde{\Pi}\) and on \(\text{Cat}(\tilde{G}, \tilde{\Pi})\); we will describe the action explicitly in the next section. Thus we get that forgetting the \(G\)-equivariance, \(B\text{Cat}(\tilde{G}, \tilde{\Pi}) \simeq E\Pi\). Therefore \(p: E(G, \Pi_G) \to B(G, \Pi_G)\) is a universal principal \(\Pi\)-bundle nonequivariantly.

In section 3.3.2 we show that \(E(G, \Pi_G)\) has the equivariant homotopy type of the to-
tal space of a universal \((G, \Pi_G)\)-bundle by showing that it satisfies the criterion given by Theorem 3.2.4. This will imply that \(p: E(G, \Pi_G) \to B(G, \Pi_G)\) is not only a nonequivariant universal principal \(\Pi\)-bundle, but that it is a universal \((G, \Pi_G)\)-bundle.

In section 3.3.3 we find a categorical model for \(B(G, \Pi_G)\). More precisely we show that

\[
B(G, \Pi_G) \simeq B\text{Cat}(\tilde{G}, \Pi),
\]

where the topological group \(\Pi\) is regarded as a topological category with a single object. For this we need to check two commutations with passage to orbits under \(\Pi\), and for some of the unenlightening details we will refer the reader to [GMM].

### 3.3.2 Categorical model for the total space

First we establish the following proposition.

**Proposition 3.3.1.** The action of \(\Pi\) on the space \(E(G, \Pi_G) = B\text{Cat}(\tilde{G}, \tilde{\Pi})\) extends to an action of \(\Pi \rtimes G\).

This will follow immediately from Lemma 3.3.2 since the classifying space of a \(\Pi \rtimes G\)-category is a \(\Pi \rtimes G\)-space. Recall that if \(\mathcal{A}\) is a \(G\)-category, then there is a left action of \(G\) on \(\text{Cat}(\mathcal{A}, \tilde{\Pi})\) given by conjugation, \((gF)(a) = g \cdot F(g^{-1}a)\) for \(g \in G\) and an object or morphism \(a \in \mathcal{A}\). There is also a right action of \(\Pi\) on \(\text{Cat}(\tilde{G}, \tilde{\Pi})\) given by \((F\sigma)(a) = F(a)\sigma\).

**Lemma 3.3.2.** For a \(G\)-category \(\mathcal{A}\), the left \(G\) and right \(\Pi\)-actions on \(\text{Cat}(\mathcal{A}, \tilde{\Pi})\) extend naturally to a \(\Gamma\)-action, where \(\Gamma = \Pi \rtimes G\).

**Proof.** By Lemma 3.2.2 it suffices to verify that \(g(F\sigma) = (gF)(g \cdot \sigma)\) for \(g \in G\), \(\sigma \in \Pi\) and a functor \(F: \mathcal{A} \to \tilde{\Pi}\). The unique natural transformation \(E \to F\) between a pair of functors \(E\) and \(F\) will then necessarily be given by \(\Gamma\)-maps. The verification is formal from the fact that \(G\) acts by conjugation, so that the action of \(G\) on \(\Pi\) is part of the prescription of the
action of $G$ on $F$. We have

$$(g(F\sigma))(a) = g \cdot (F\sigma)(g^{-1}a)$$

$$= g \cdot (F(g^{-1}a)\sigma)$$

$$= (g \cdot F(g^{-1}a))(g \cdot \sigma)$$

$$= ((gF)(a))(g \cdot \sigma)$$

$$= ((gF)(g \cdot \sigma))(a). \quad \square$$

**Corollary 3.3.3.** The map $p: E(G, \Pi_G) \to B(G, \Pi_G)$ is a principal $(G, \Pi_G)$-bundle.

*Proof.* Observe that the action of $\Pi$ on $\text{Cat}(\tilde{G}, \tilde{\Pi})$ is free, and therefore, since the classifying space functor commutes with fixed points, the action on the classifying space of this category, namely on $E(G, \Pi_G)$, is also free. Thus $p: E(G, \Pi_G) \to B(G, \Pi_G) = E(G, \Pi_G)/\Pi$ is a principal $\Pi$-bundle, and by Proposition 3.3.1 we have that it is a principal $(G, \Pi_G)$-bundle.

Now we go on to show that the $\Pi \rtimes G$-space $E(G, \Pi_G)$ is the total space of a universal principal $(G, \Pi_G)$-bundle.

**Theorem 3.3.4.** If $\Pi$ is a topological group, and $G$ is a discrete group acting on $\Pi$, the principal $(G, \Pi_G)$-bundle $p: E(G, \Pi_G) \to B(G, \Pi_G)$ is universal.

By the criterion for universal bundles given in Theorem 3.2.4, it suffices to show that for $\Lambda \subseteq \Pi \rtimes G$ such that $\Lambda \cap \Pi = e$, the fixed point spaces $E(G, \Pi_G)^\Lambda$ are contractible.

We give a convenient description of the subgroups $\Lambda \subseteq \Pi \rtimes G$ with the property that $\Lambda \cap \Pi = e$. Recall Definition 2.5.1 of a crossed homomorphism.

**Lemma 3.3.5.** All subgroups $\Lambda$ of $\Gamma$ such that $\Lambda \cap \Pi = e$ are of the form

$$\Lambda_\alpha = \{(\alpha(h), h) | h \in H\},$$

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where $H$ is a subgroup of $G$ and $\alpha: H \to \Pi$ is a crossed homomorphism. At least if $G$ is discrete or $\Gamma$ is compact, $\alpha$ is continuous.

**Proof.** Clearly $\Lambda_\alpha$ is a subgroup of $\Gamma$ such that $\Lambda_\alpha \cap \Pi = e$. Conversely, let $\Lambda \cap \Pi = e$. Define $H$ to be the image of the composite of the inclusion $\iota: \Lambda \subset \Gamma$ and the projection $\pi: \Gamma \to G$. Since $\Lambda \cap \Pi = e$, the composite $\pi \circ \iota$ is injective and so restricts to a continuous isomorphism $\nu: \Lambda \to H$. For $h \in H$, define $\alpha(h) = \sigma$, where $\sigma$ is the unique element of $\Pi$ such that $(\sigma, h) \in \Lambda$. Thus $\alpha$ is the composite of $\iota \circ \nu^{-1}: H \to \Gamma$ and the projection $\rho: \Gamma \to \Pi$. If $G$ is discrete or if $\Gamma$ and therefore $\Lambda$ is compact, then $\nu$ is a homeomorphism and $\alpha$ is continuous.

For $h,k \in H$,

$$(\alpha(h), h)(\alpha(k), k) = (\alpha(h)(h \cdot \alpha(k)), hk) \in \Lambda,$$

so $\alpha(hk) = \alpha(h)(h \cdot \alpha(k))$. Thus $\alpha$ is a crossed homomorphism and $\Lambda = \Lambda_\alpha$. \hfill $\square$

**Proof of Theorem 3.3.4.** Note that since there is a unique morphism between any two objects in $\text{Cat}(\tilde{G}, \tilde{\Pi})$, the same is true of the fixed point subcategory $\text{Cat}(\tilde{G}, \tilde{\Pi})^\Lambda$ if this is nonempty. Thus every object in this full subcategory is again initial and terminal, and $B\text{Cat}(\tilde{G}, \tilde{\Pi})^\Lambda$ is contractible.

Therefore, if we show that $\text{Cat}(\tilde{G}, \tilde{\Pi})^\Lambda$ is non-empty for all $\Lambda \subseteq \Pi \times G$ such that $\Lambda \cap \Pi = e$, the statement of the theorem follows by Theorem 3.2.4.

Upon taking fixed points, the adjunction homeomorphism 2.2 from Remark 2.3.5 gives

$$\text{Cat}(\tilde{G}, \tilde{\Pi})^\Lambda \cong \text{Map}(G, \Pi)^\Lambda.$$

Therefore we must exhibit a $\Lambda$-map $f: G \to \Pi$, where $\Lambda = \Lambda_\alpha$ for a crossed homomorphism $\alpha$. By the definition of the action by $\Lambda$, this means that

$$f(g) = (h \cdot f(h^{-1}g)) \alpha(h)^{-1}$$
or equivalently

\[ h \cdot f(h^{-1}g) = f(g) \alpha(h) \]

for all \( h \in H \) and \( g \in G \). We choose right coset representatives \( \{g_i\} \) to write \( G \) as a disjoint union of cosets \( Hg_i \). We then define \( f : G \to \Pi \) by

\[ f(kg_i) = \alpha(k)^{-1} \]

for \( k \in H \). By using (2.7), writing out the inverse of a product as the product of inverses, using that \( h^{-1} \) and \( h \cdot \) are group homomorphisms and that \( \cdot \) is a group action, and finally using (2.8) and, again, that \( \cdot \) is a group action, we see that

\[
\begin{align*}
  h \cdot f(h^{-1}kg_i) & = h \cdot (\alpha(h^{-1})(h^{-1} \cdot \alpha(k))^{-1} \\
                         & = h \cdot ((h^{-1} \cdot \alpha(k))^{-1}(\alpha(h^{-1}))^{-1}) \\
                         & = (h \cdot (h^{-1} \cdot \alpha(k)^{-1})(h \cdot (\alpha(h^{-1})^{-1}) \\
                         & = \alpha(k)^{-1}(h \cdot (h^{-1} \cdot \alpha(h))) \\
                         & = f(kg_i) \alpha(h).
\end{align*}
\]

for all \( h \in H \). Thus \( f \) is a \( \Lambda \)-map. We have assumed that \( G \) is discrete in order to ensure that \( f \) is continuous.

\[ \square \]

### 3.3.3 Categorical model for the classifying space

Now that we have shown that

\[ p: B\text{Cat}(\tilde{G}, \tilde{\Pi}) \to B\text{Cat}(\tilde{G}, \tilde{\Pi})/\Pi \]
is a universal principal \((G, \Pi_G)\)-bundle, we give a categorical interpretation of the classifying space \(BCat(\tilde{G}, \tilde{\Pi})/\Pi\) for \((G, \Pi_G)\)-bundles.

Regard the topological group \(\Pi\) as a topological category with a single object \(*\) and morphism space \(\Pi\). Recall that \(\Pi\) acts from the right on \(\tilde{\Pi}\) via right multiplication on objects and diagonal right multiplication on morphisms, and note that \(\Pi\) is isomorphic to the orbit category \(\tilde{\Pi}/\Pi\). The quotient functor \(q: \tilde{\Pi} \to \Pi\) is given by the trivial map \(\Pi \to *\) of object spaces and the map \(\Pi \times \Pi \to \Pi \times \Pi/\Pi \cong \Pi\) on morphism spaces which sends \((\tau, \sigma)\) to \(\tau\sigma^{-1}\). This is a \(G\)-map since

\[
g \cdot q(\tau, \sigma) = g \cdot (\tau\sigma^{-1}) = (g \cdot \tau)(g \cdot \sigma)^{-1} = q(g \cdot \tau, g \cdot \sigma).
\]

By applying the functors \(Cat(\tilde{G}, -)\) and \(B\) to the map \(q\), we get a \(G\)-map

\[
p: BCat(\tilde{G}, \tilde{\Pi}) \to BCat(\tilde{G}, \Pi),
\]
and the claim is that it is equivalent to passage to orbits over \(\Pi\).

**Theorem 3.3.6.** There is a \(G\)-homotopy equivalence

\[
BCat(\tilde{G}, \tilde{\Pi})/\Pi \simeq BCat(\tilde{G}, \Pi).
\]

For the details of the proof, we refer the reader to [GMM]. The proof breaks down into the following two steps, neither of which is very difficult to show.

**Step 1.** The first step is to show that quotienting out the action of \(\Pi\) commutes with the classifying space functor. Note that the nerve functor \(N\) is a right adjoint, so it does not in general commute with colimits, and it is not usually the case that \(N(\mathcal{C}/G) \cong (N\mathcal{C})/G\) for a \(G\)-category \(\mathcal{C}\), as the following counterexample should make clear.

**Example 3.3.7.** Let \(G\) be a group and let \(G\) act on itself by conjugation. Let \(A\) be the
abelianization of $G$. Regarding $G$ and $A$ as categories with a single object, $G/G \cong A$, and $NA$ is generally much smaller than $NG/G$. Here $[g_1, \ldots, g_q]$ and $[h_1, \ldots, h_q]$ are in the same orbit under the conjugation action if and only if there is a single $g$ such that $gg_ig^{-1} = gh_ih^{-1}$ for all $i$. For example if $G$ is a finite simple group of order $n$, then $A$ is trivial but $N_qG/G$ has at least $n^{q-1}$ elements.

However, in the case we are considering we do obtain the desired commutation relation. It is a more general manifestation of the phenomenon $(EG)/G = BG$ where, on the right, $G$ is regarded as a one object groupid. Equivalently,

$$(B\tilde{G})/G \simeq B(\tilde{G}/G).$$

More generally,

$$(B\tilde{G})/H \simeq B(\tilde{G}/H)$$

for a subgroup $H \subseteq G$. The left hand side is equivalent to $EG/H$, which is $BH$ since $EG$ is an $H$-free contractible space. On the right hand side, $\tilde{G} \simeq \tilde{H}$ as $H$-categories, so that $\tilde{G}/H \simeq \tilde{H}/H = H$. Thus right hand side is therefore also a model for $BH$.

The essential point for us is that the category $\text{Cat}(\tilde{G}, \tilde{\Pi})$ has object space the group $\text{Map}(G, \Pi)$ with group operation induced from $\Pi$, and there is a unique morphism between any two objects. Thus $\text{Cat}(\tilde{G}, \tilde{\Pi})$ can be identified with $\text{Map}(\tilde{G}, \Pi)$, and $B\text{Cat}(\tilde{G}, \tilde{\Pi})$ is equivalent to $E\text{Map}(G, \Pi)$. Now the quotient by the $\Pi$-action corresponds to the quotient by the subgroup $\Pi$ of $\text{Map}(G, \Pi)$. Thus applying the phenomenon described in the previous paragraph we get that

$$(B\text{Cat}(\tilde{G}, \tilde{\Pi}))/\Pi \simeq B(\text{Cat}(\tilde{G}, \tilde{\Pi})/\Pi).$$

It is not hard to see that this is a $G$-equivalence.

**Step 2.** The second step is to show that we have a second commutation relation with
respect to the quotient by \( \Pi \), namely that

\[
\text{Cat}(\tilde{G}, \tilde{\Pi})/\Pi \simeq \text{Cat}(\tilde{G}, \Pi).
\]

This is verified explicitly in [GMM, 2.9]. Again, it is not hard to see that this is \( G \)-equivariant; the details of the compatibility with the \( G \)-action are in [GMM, 3.4., 3.5.].

We end by saying that there are simple, categorical descriptions for the fixed points of the classifying spaces for \((G, \Pi G)\)-bundles arising from the interpretation of the homotopy fixed point category of a group \( \Pi \), which was given in section 2.5. We refer the reader to [GMM] for a comprehensive categorical treatment of the fixed points of classifying spaces of \((G, \Pi G)\)-bundles and a comparison with the pre-existing space-level descriptions.

### 3.3.4 Conclusion

We combine theorems Theorem 3.3.4 and Theorem 3.3.6 into one statement.

**Theorem 3.3.8.** The natural map

\[
p: B\text{Cat}(\tilde{G}, \tilde{\Pi}) \to B\text{Cat}(\tilde{G}, \Pi)
\]

induced by the quotient \( q: \tilde{\Pi} \to \Pi \) is a universal principal \((G, \Pi G)\)-bundle.

We come back to the motivations outlined in section 3.1 to say what this result accomplishes.
1. Equivariant infinite loop space theory

In section 3.1.1 we explained how having a categorical model for a universal principal \((G, \Sigma_j)\)-bundle leads to the definition of an \(E_\infty\)-operad in \(G\text{Cat}\). We reemphasize that in these bundles \(G\) acts trivially on \(\Sigma_j\). We repeat the definition of an \(E_\infty\)-operad ([GMa, 1.11.]).

**Definition 3.3.9.** An operad \(\mathcal{O}_G\) in \(G\text{Top}\) is \(E_\infty\) if \(\mathcal{O}_G(j)\) is the total space of a universal principal \((G, \Sigma_j)\)-bundle. An operad \(\mathcal{O}_G\) in \(G\text{Cat}\) is \(E_\infty\) if the operad \(B\mathcal{O}_G\) in \(G\text{Top}\) is \(E_\infty\).

Thus we get the following immediate consequence of Theorem 3.3.4.

**Proposition 3.3.10.** The operad \(\mathcal{O}_G\) in \(G\text{Cat}\) with \(\mathcal{O}_G(j) = \text{Cat}(\tilde{G}, \tilde{\Sigma}_j)\) is \(E_\infty\).

As is pointed out in [GMa], the bundle theory dictates the homotopical properties of \(\mathcal{O}_G\), and it is these characterizing properties that are relevant to equivariant infinite loop space theory, not their bundle theoretic consequences.

2. Equivariant algebraic \(K\)-theory

In section 3.1.2 we explained how having a categorical model for the classifying space of \((G, GL_n(R))\)-bundles, when \(R\) is a \(G\)-ring, leads to the definition of equivariant algebraic \(K\)-theory. For \(i \geq 1\), Quillen defined the algebraic \(K\)-theory groups \(K_i(R)\) as the homotopy groups of the group completion of \(\bigsqcup_n BGL_n(R)\), which is a topological monoid with operation given by Whitney sum of bundles. Given Theorem 3.3.6, the intuitive equivariant generalization is to define the equivariant algebraic \(K\)-theory groups \(K^H_i(R)\) of a \(G\)-ring \(R\), for \(i \geq 1\), as the equivariant homotopy groups of the equivariant group completion of the topological \(G\)-monoid \(\bigsqcup_n BCat(\tilde{G}, GL_n(R))\). This is the starting point of equivariant higher algebraic \(K\)-theory developed in the next chapters, and we will come back to the idea described above in Section 4.3.2. We will give the definition of topological group completion and its equivariant counterpart in the next chapter. Equivariant homotopy groups will also be formally defined in the next chapter.
CHAPTER 4
EQUIVARIANT K-THEORY OF G-RINGS

4.1 Equivariant group completion

4.1.1 Group completion in K-theory

The idea behind defining the zeroth $K$ group was to strengthen the structure of an abelian monoid by formally adding inverses and thus forming its group completion. The most basic example is that of the monoid $\mathbb{N}$ whose group completion is $\mathbb{Z}$. In general, for an abelian monoid $M$ we define the Grothendieck group of $M$, denoted $\text{Gr}(M)$, to be the free abelian group on generators $[m]$, for $m \in M$, modulo the subgroup generated by elements of the form $[m] + [n] - [m + n]$, and define the group completion map $M \to \text{Gr}(M)$ to be the map that sends each $m$ to the class $[m]$. This map has the universal property that every monoid homomorphism from $M$ to an abelian group factors through it.

The $K_0$ groups were introduced by Grothendieck around 1956 in the context of sheaves over algebraic varieties. Inspired by Grothendieck’s work, M. Atiyah and F. Hirzebruch introduced topological $K$-theory in 1959. They defined the $K^0$ group of a finite CW complex $X$ as the Grothendieck group of the abelian monoid $\text{Vect}(X)$ of isomorphism classes of finite-dimensional vector bundles (real or complex) over $X$ with the Whitney sum of vector bundle operation:

$$KU^0(X) = \text{Gr}(\text{Vect}_{\mathbb{C}}(X)) \quad \text{and} \quad KO^0(X) = \text{Gr}(\text{Vect}_{\mathbb{R}}(X)).$$

This motivates the following algebraic definition of the group $K_0(R)$, where $R$ is a ring with unity.

**Definition 4.1.1.** Let $\mathcal{P}(R)$ denote the abelian monoid of isomorphism classes of finitely generated projective $R$-modules. Then $K_0(R)$ is the Grothendieck group $\text{Gr}(\mathcal{P}(R))$. 

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For example, if $R$ is a field, a PID, or a local ring, then every finitely generated projective module is free, and $K_0(R) \cong \mathbb{Z}$. When $R$ is a number ring, we can deduce from the classification theorem of finitely generated projective modules over Dedekind domains that $K_0(R) = \mathbb{Z} \oplus \text{Pic}(R)$, which makes the $K_0$ group relevant in number theory. There are a lot of applications of $K_0$ in geometric topology also, but we will not diverge from our story to go into the details of those here. A great reference of an application of the group $K_0$ of a ring to geometric topology is Wall's paper [Wal65].

The higher algebraic $K$-theory is defined as the group completion of a topological (as opposed to just an algebraic) monoid, which is formed by coherently adjoining homotopy inverses. Since the concept of group completion is central to this chapter, we briefly review the nonequivariant definitions and spell out the equivariant analogues in the next section.

### 4.1.2 Equivariant group completion of a Hopf $G$-space

Recall that a homotopy associative, homotopy commutative Hopf space $X$ is a topological space with a multiplication that is associative and commutative up to homotopy, and an identity element in the sense that multiplication by this element is homotopic to the identity. Its set of path components $\pi_0(X)$ is an abelian monoid, $H_0(X; \mathbb{Z})$ is the monoid ring $\mathbb{Z}[\pi_0(X)]$, and the integral homology $H_*(X; \mathbb{Z})$ is an associative graded-commutative ring with unit.

For example, if $S$ is a symmetric monoidal category, i.e., a category with an operation $\oplus : S \times S \to S$ which is commutative, associative and with a distinguished identity element, all up to coherent natural isomorphism, then $BS$ is a homotopy commutative, homotopy associative Hopf space with multiplication induced by $\oplus$.

**Note 4.1.2.** We decided not to use the standard shorthand name “$H$-space” for a Hopf space in order to avoid utter confusion with the notion of $H$-space in the sense of having a group action of the group $H$ on the space. Since we are using the latter notion heavily in
this manuscript, we will refrain from using the same name for anything else.

What we mean by group completion is the following, and the motivation behind this construction comes from the group completion of an abelian monoid discussed above:

**Definition 4.1.3.** A group completion of a homotopy associative and commutative Hopf space $X$ is a homotopy associative and commutative Hopf space $Y$, together with an Hopf space map $X \to Y$, such that

$$\pi_0(X) \to \pi_0(Y)$$

is a group completion of abelian monoids, and the ring map

$$H_*(X, R) \to H_*(Y, R)$$

is localization with respect to the multiplicatively closed subset $\pi_0(X)$ for all commutative rings $R$.

A Hopf $G$-space is a Hopf space with equivariant multiplication map and for which multiplying by the identity element is $G$-homotopic to the identity map. For example, if $X$ is a $G$-space, then $\Omega X$ is a Hopf $G$-space. The equivariant notion of group completion is captured by the fixed point maps being group completions.

**Definition 4.1.4.** A $G$-map $X \to Y$ of homotopy associative and commutative Hopf $G$-spaces is an equivariant group completion if the fixed point maps $X^H \to Y^H$ are group completions for all $H \subseteq G$.

For a homotopy commutative topological monoid there is a model for the group completion, which is very well-known in the case of a topological group, where it yields an equivalence:

**Theorem 4.1.5** (Group completion theorem, [MS76], [May75]). *If $M$ is a homotopy commutative topological monoid, the map $M \to \Omega BM$ is a group completion.*
Since the classifying space functor $B$ and the loop functor $\Omega$ have the wonderful virtue of commuting with fixed points, we immediately get the following consequence.

**Corollary 4.1.6.** If $M$ is a homotopy commutative topological $G$-monoid, the map $M \to \Omega BM$ is an equivariant group completion.

### 4.1.3 The equivariant group completion of the classifying space of a symmetric monoidal $G$-category

A *symmetric monoidal $G$-category* is a symmetric monoidal category $\mathcal{C}$ with $G$-action which commutes with the symmetric monoidal structure. In other words, it is a functor $G \to \text{SymCat}$, the category of symmetric monoidal categories and strict functors. Note that the classifying space $B\mathcal{C}$ of a symmetric monoidal $G$-category is a Hopf $G$-space. We give a functorial construction of the group completion of $B\mathcal{C}$, following [Qui73]. The idea is to define the group completion on the category level.

First, we give the model for the categorical group completion in the nonequivariant case, and then we observe that the theory carries through equivariantly.

**Definition 4.1.7.** Let $S$ be a symmetric monoidal category. We define the category $S^{-1}S$ to have objects pairs $(m, n)$ of objects in $S$. A morphism $(m, n) \to (p, q)$ in $S^{-1}S$ is an equivalence class of triples

$$(r, r \oplus m \xrightarrow{f} p, r \oplus n \xrightarrow{g} q)$$

where this triple is equivalent to the triple

$$(r', r' \oplus m \xrightarrow{f'} p, r' \oplus n \xrightarrow{g'} q)$$

if there is an isomorphism $r \cong r'$ making the relevant diagram commute. Composition for a
pair of morphisms

\[(r, r \oplus m \xrightarrow{f} p, r \oplus n \xrightarrow{g} q)\]

and

\[(s, s \oplus p \xrightarrow{\phi} u, s \oplus q \xrightarrow{\psi} v)\]

is given by

\[(s \oplus r, s \oplus r \oplus m \xrightarrow{\phi \circ (s \oplus f)} u, s \oplus r \oplus n \xrightarrow{\psi \circ (s \oplus g)} v).\]

Note that \(S^{-1}S\) is symmetric monoidal with \((m, n) \oplus (p, q) = (m \oplus p, n \oplus q)\). We have a strict monoidal functor \(S \to S^{-1}S\) given by \(m \mapsto (m, 0)\), where 0 is the unit of \(S\). This induces a map of Hopf spaces

\[BS \to BS^{-1}S,\]

which in turn induces a map of abelian monoids

\[\pi_0(BS) \to \pi_0(BS^{-1}S).\]

It is not hard to see that \(\pi_0(BS^{-1}S)\) is actually an abelian group; the inverse of an element representing the object \((m, n)\) in \(\pi_0\) is \((n, m)\). It turns out that the map (4.1) is a group completion of Hopf spaces. The original proof is in [Qui73], and a great detailed exposition can be found in Guillou’s note [Gui].

Note that if \(S\) had an initial or terminal object, then \(BS\) would be contractible. Unless \(S\) is a symmetric monoidal groupoid, we restrict our attention to the category of isomorphisms \(\text{iso}S\). This is still symmetric monoidal, so \(B(\text{iso}S)\) is a Hopf space.

**Theorem 4.1.8** ([Qui73]). *Let \(S\) be a symmetric monoidal groupoid such that translations are faithful, i.e.,

\[\text{Aut}(s) \to \text{Aut}(s \oplus t)\]
is injective for all \( s, t \in S \). Then the map \( BS \to BS^{-1}S \) is a group completion.

Now if \( S \) is a symmetric monoidal \( G \)-category, then \( S^{-1}S \) is also a symmetric monoidal \( G \)-category with diagonal action on objects. On morphisms,

\[
g((m, n) \xrightarrow{(r, f, f')} (p, q)) = (gm, gn) \xrightarrow{(gr, gf, gf')} (gp, gq).
\]

Note that this only works, because the action of \( G \) commutes with \( \oplus \). The fixed point subcategory \( S^H \) is also a symmetric monoidal category, thus we can form \((S^H)^{-1}(S^H)\), and it is not hard to see that the construction commutes with fixed points.

**Lemma 4.1.9.** Let \( S \) be a symmetric monoidal \( G \)-category. Then

\[
(S^H)^{-1}(S^H) \cong (S^{-1}S)^H
\]

for all \( H \subseteq G \).

Also, note that if translations are faithful in \( S \), i.e., if \( \text{Aut}(s) \to \text{Aut}(s \oplus t) \) is injective for all \( s, t \in S \), then the same holds for the fixed point subcategories \( S^H \). This has the following immediate consequence.

**Proposition 4.1.10.** Let \( S \) be a symmetric monoidal \( G \)-groupoid such that translations are faithful. Then the map \( BS \to BS^{-1}S \) is an equivariant group completion.

Note that if \( S \) is a symmetric monoidal \( G \)-category, then so is \( \text{Cat}(\tilde{G}, S) \), with

\[
\oplus: \text{Cat}(\tilde{G}, S) \times \text{Cat}(\tilde{G}, S) \to \text{Cat}(\tilde{G}, S)
\]

given by

\[
(F \oplus G)(x) = F(x) \oplus G(x),
\]
and identity element given by the constant functor at 0, the identity of $S$. We check that the action on $\text{Cat}(\tilde{G}, S)$ commutes with $\oplus$:

$$g(F \oplus G)(g^{-1}x) = g(F(g^{-1}x) \oplus G(g^{-1}x)) = gF(g^{-1}x) \oplus gG(g^{-1}x).$$

We note the following useful result about restricting to the subcategory of isomorphisms.

**Lemma 4.1.11.** For any $G$-category $\mathcal{C}$, we have an identification

$$\text{iso } \text{Cat}(\tilde{G}, \mathcal{C}) \cong \text{Cat}(\tilde{G}, \text{iso } \mathcal{C}).$$

**Proof.** Note that a functor $F : \tilde{G} \to \mathcal{C}$ actually lands in $F : \tilde{G} \to \text{iso } \mathcal{C}$ since every morphism in $\tilde{G}$ is an isomorphism. Therefore the objects of $\text{iso } \text{Cat}(\tilde{G}, \mathcal{C})$ and $\text{Cat}(\tilde{G}, \text{iso } \mathcal{C})$ are the same. Now a morphism in $\text{iso } \text{Cat}(\tilde{G}, \mathcal{C})$ is a natural transformation whose component maps are all isomorphisms, which is the same with a morphism in $\text{Cat}(\tilde{G}, \text{iso } \mathcal{C})$. \hfill $\square$

### 4.2 Equivariant $K$-theory space

#### 4.2.1 Equivariant $K$-theory space and equivariant $K$-groups

Now that we have the functorial model for the group completion of a symmetric monoidal $G$-category given in Proposition 4.1.10 and the identification given in Lemma 4.1.11, we make the following definition, which generalizes the nonequivariant one. In the next section, we will interpret it in terms of bundle theory.

**Definition 4.2.1.** The equivariant algebraic $K$-theory space of a symmetric monoidal $G$-category $S$ is $K_G(S) = B(\text{Cat}(\tilde{G}, \text{iso } S)^{-1}\text{Cat}(\tilde{G}, \text{iso } S))$.

For a $G$-ring $R$, we make the definition:
**Definition 4.2.2.** The equivariant algebraic $K$-theory space of a $G$-ring $R$ is

$$K_G(R) = K_G(\mathcal{P}(R)).$$

By Lemma 2.7.20, we immediately get Morita invariance.

**Proposition 4.2.3.** If $R$ and $S$ are equivariantly Morita equivalent, then there is a $G$-equivalence

$$K_G(R) \simeq K_G(S).$$

We define the equivariant $K$-groups as the equivariant homotopy groups of this space. Recall that for a subgroup $H \subseteq G$ and a $G$-space $X$, we have

$$\pi^H_i(X) = [(G/H)_+, \wedge S^i, X]_G = [S^i, X^H] = \pi_i(X^H),$$

where $[X,Y]_G$ denotes the set of homotopy classes of based $G$-maps $X \to Y$ between based $G$-spaces, and $X_+$ denotes the union of $X$ with a disjoint basepoint. We define the $K$-groups for $i \geq 0$.

**Definition 4.2.4.** The algebraic $K$-theory groups are given by

$$K_i^H(R) = \pi^H_i(K_G(R)).$$

This definition will be reconciled with the intuitive definition suggested in Section 3.3.4 in the next section.

### 4.2.2 Equivariant $K_0$

Recall that $\mathcal{P}(R)$ is the category of finitely generated projective modules over $R$, so that the abelian monoid $\mathcal{P}(R)$, which figures in Definition 4.1.1, is the abelian monoid of isomorphism
classes of objects in the symmetric monoidal category $\mathcal{P}(R)$. Therefore,

$$\mathcal{P}(R) \cong \pi_0(B \text{ iso } \mathcal{P}(R)). \quad (4.3)$$

Since $K_0(R)$ is the group completion of the abelian monoid $\mathcal{P}(R)$, we can reinterpret it using Definition 4.1.3 and (4.3) as $\pi_0$ of the group completion of $B(\text{ iso } \mathcal{P}(R))$.

We spell out what the equivariant $K_0$ is so that it is clear how it relates back to the nonequivariant one via the above interpretation. From Definition 4.2.4 and (4.2), we have

$$K^H_0(R) = \pi_0(K_G(R)^H).$$

From Definition 4.2.2 we have that $K_G(R)^H$ is the group completion of $B\text{Cat}(\tilde{G}, \text{ iso } \mathcal{P}(R))^H$. Therefore $K^H_0(R)$ is the group completion of the monoid of isomorphism classes of objects in the category $\text{ iso } \mathcal{P}(R)^{hH} = \text{Cat}(\tilde{G}, \text{ iso } \mathcal{P}(R))^H$.

### 4.3 Connection to equivariant bundle theory

#### 4.3.1 Fiedorowicz-Hauschild-May definition

Equivariant higher $K$-groups for ring a $R$ with trivial $G$-action were first defined in 1982 by Fiedorowicz, Hauschild and May in [FHM82] via an equivariant version of the plus construction, interpreted appropriately. They used this definition to prove an equivariant Adams conjecture and compute the equivariant $K$-theory of finite fields with trivial $G$-action. Whereas there are interesting examples of equivariant $K$-theory arising from rings on which $G$ acts trivially, such as equivariant topological $K$-theory $ku_G$ or $ko_G$, the most interesting examples for us will arise from nontrivial action on the input ring.

The main idea behind the definition in [FHM82] is to replace use of the plus construction of $BGL(R)$ by the group completion $\Omega B\mathcal{G}\mathcal{L}(R)$ of the classifying space of permutative
category \(\mathcal{GL}(R)\). However, the given definition assumes that the group \(G\) acts trivially on \(GL_n(R)\), and the bundles implicitly considered are universal principal \((G, GL_n(R))\)-bundles with their classifying spaces. Their models for the classifying spaces are also much more unwieldy than the categorical ones we have worked out in chapter 3.

First we will generalize the definition of Fiedorowicz, Hauschild and May so as to allow nontrivial action of \(G\) on the ring \(R\). Instead of using the classifying spaces of equivariant \((G, GL_n(R))\)-bundles, which correspond to a trivial group extension

\[
1 \rightarrow GL_n(R) \rightarrow GL_n(R) \times G \xrightarrow{q} G \rightarrow 1,
\]

we will use the classifying spaces of \((G, GL_n(R)_G)\)-bundles, which correspond to split extensions

\[
1 \rightarrow GL_n(R) \rightarrow GL_n(R) \rtimes G \xrightarrow{q} G \rightarrow 1.
\]

Suitable categorical models for universal \((G, GL_n(R)_G)\)-bundles have been given in section 3.2, and these are central to our definition.

Second, we show that this yields the same equivariant higher homotopy groups as the \(K\)-theory space introduced in Definition 4.2.2.

4.3.2 An equivalent construction in terms of equivariant bundles

As we suggested at the end of chapter 3, we want the higher \(K\)-theory groups to be the equivariant homotopy groups of the group completion of the \(G\)-monoid given by the disjoint union of classifying spaces of \((G, GL_n(R)_G)\)-bundles. Recall from Definition 2.7.16 that \(\mathcal{GL}(R) = \coprod_n GL_n(R)\). By Theorem 3.3.8, this monoid is \(\mathcal{B}Cat(\tilde{G}, \mathcal{GL}(R))\), and a model for the group completion is, by Corollary 4.1.6,

\[
\Omega B\mathcal{B}Cat(\tilde{G}, \mathcal{GL}(R)).
\]
We proceed to show that we have an equivalence

\[ K^H_G(R) \simeq \Omega B B \text{Cat}(\tilde{G}, \mathcal{G}(R))^H \]

on basepoint components, so these spaces have the same higher homotopy groups. This shows that the definition of [FHM82] of higher equivariant \( K \)-groups for a ring with trivial \( G \)-action agrees with the one given in Definition 4.2.4.

Again, we will follow Quillen’s nonequivariant proof. We recall the definition of cofinality and then state the result that leads to showing that cofinality gives an equivalence on higher \( K \)-theory. The proof is in [Qui73] and a great exposition is also given in [Gui].

**Definition 4.3.1.** A monoidal functor \( F : S \to T \) is cofinal if for every \( t \in T \) there is \( t' \in T \) and \( s \in S \) such that \( t \oplus t' = F(s) \).

**Proposition 4.3.2 ([Qui73]).** If \( F : S \to T \) is cofinal and \( \text{Aut}_S(s) \cong \text{Aut}_T(F(s)) \) for all \( s \in S \), then the map \( B(S^{-1}S) \to B(T^{-1}T) \) induces an equivalence of basepoint components.

Nonequivariantly this is applied to the cofinal inclusion \( \text{iso } F(R) \hookrightarrow \text{iso } \mathcal{P}(R) \), which is in fact the idempotent completion. We wish to apply it to the fixed point maps of the inclusions

\[ \text{Cat}(\tilde{G}, \text{iso } F(R))^H \to \text{Cat}(\tilde{G}, \text{iso } \mathcal{P}(R))^H. \]

For this we recall from section 2.7.3 that the homotopy fixed point category \( \text{Cat}(\tilde{G}, \text{iso } F(R))^H \) is the category of finitely generated free \( R \)-modules with semilinear \( H \)-action and \( H \)-isomorphism. Similarly, \( \text{Cat}(\tilde{G}, \text{iso } \mathcal{P}(R))^H \) is the category of finitely generated projective \( R \)-modules with semilinear \( H \)-action. This inclusion is cofinal; however it is not an idempotent completion anymore. One way to see this is to consider the case when \( G \) is finite and \( |G|^{-1} \in R \), so that, by Corollary 2.7.15, \( \text{Cat}(\tilde{G}, \text{iso } \mathcal{P}(R))^H \simeq \mathcal{P}(R_H[H]) \). We have a commutative diagram of inclusions:
The diagonal map is an idempotent completion, but the map going straight up is not an equivalence, since free $R_H[H]$-modules do not coincide with modules with semilinear $G$-action which are free as $R$-modules. Therefore, the top map is not the idempotent completion; it just factors through it.

**Theorem 4.3.3.** There is an equivalence on connected basepoint components

$$\Omega B B \text{Cat}(\tilde{\mathcal{G}}, \text{iso } \mathcal{F}(R))^H \simeq_0 K_G(R)^H.$$  

(We used the notation $\simeq_0$ instead of $\simeq$ in order to emphasize that this equivalence only holds on basepoint components and to avoid possible misinterpretation.)

**Proof.** By Proposition 2.7.17, the inverse of the nonequivariant equivalence given by the inclusion of the skeleton $i: \mathcal{L}(R) \to \text{iso } \mathcal{F}(R)$ induces a weak $G$-equivalence

$$\text{Cat}(\tilde{\mathcal{G}}, \text{iso } \mathcal{F}(R)) \to \text{Cat}(\tilde{\mathcal{G}}, \mathcal{L}(R)).$$

Now

$$\text{Cat}(\tilde{\mathcal{G}}, \text{iso } \mathcal{F}(R))^H \longrightarrow \text{Cat}(\tilde{\mathcal{G}}, \text{iso } \mathcal{P}(R))^H$$

is cofinal. Therefore, by applying Proposition 4.3.2, we get a weak $G$-equivalence of basepoint components

$$B(S^{-1}S) \to B(T^{-1}T)$$

for $S = \text{Cat}(\tilde{\mathcal{G}}, \text{iso } \mathcal{F}(R))^H$ and $T = \text{Cat}(\tilde{\mathcal{G}}, \text{iso } \mathcal{P}(R))^H$. 

\[\square\]
Remark 4.3.4. Note that for $R = F$ a field with $G$-action, since the categories $\mathcal{F}(R)$ and $\mathcal{P}(R)$ are the same, and since we have a weak $G$-equivalence $\text{Cat}(\tilde{G}, \mathcal{L}(F)) \simeq \text{Cat}(\tilde{G}, \text{iso}\mathcal{F}(F))$, we can equivalently define the $K$-theory space of $F$ as

$$K_G(F) = \Omega B B \text{Cat}(\tilde{G}, \mathcal{L}(F)).$$

Note that in this case

$$K^H_0(F) = \pi_0(\Omega B B \text{Cat}(\tilde{G}, \mathcal{L}(F))^H) \simeq \mathbb{Z},$$

for all $H \subseteq G$.

4.3.3 Some remarks on the definition of the $K$-theory space

We note that equivariantly there is no meaningful way to write down a decomposition of the $K$-theory space as a product of $K_0$ and a connected component, analogous to the widely used nonequivariant one, which is

$$K(R) \simeq K_0(R) \times \Omega_0 BB\mathcal{L}(R).$$

The reason is that taking basepoint components does not commute with taking fixed points, so if we split off the basepoint component we change the equivariant homotopy type of the space.

However, even nonequivariantly, defining the $K$-theory space as $K(R) = K_0(R) \times \Omega_0 BB\mathcal{L}(R)$ is technically incorrect, because this is not functorial. For a proof that this decomposition cannot be made functorial, see [Sch11, 2.2.9.] In the proof of the $plus = Q$ theorem [Qui73], it is shown that this space is homotopy equivalent to the $K$-theory space defined via the $Q$-construction on the exact category of finitely generated projective modules.
The latter construction is functorial; however, there is no functorial zig-zag of equivalences between the two spaces nor is there a way to define a homotopy equivalence that respects the $H$-space structures.

The definition of the equivariant algebraic $K$-theory space we gave in section 4.2 is a better definition even nonequivariantly, because it is functorial; it appears in [Gra76] as one of the intermediary steps in the proof of $\text{plus} = Q$, which goes from the “wrong” definition as the decomposition $K_0(R) \times BGL(R)^+$ to the “right” functorial definition as $\Omega BQ\mathcal{P}(R)$.

In chapter 5, we will give definitions of the equivariant $K$-theory space in terms of the $Q$ and $S_\bullet$ constructions, and we will show that they are weakly $G$-equivalent to the space in Definition 4.2.2. However, we do not know yet how to equivariantly deloop those constructions in general. The delooping machines that have been generalized equivariantly are the group completion infinite loop space machines, which we treat in the next section. The best definition of the $K$-theory space will then be as the zeroth space of the genuine $\Omega$-$G$-spectra output by the machines.

4.4 Equivariant delooping of the $K$-theory space

4.4.1 Equivariant operadic machine

Algebras over the Barratt-Eccles operad $\mathcal{O}$ in $\text{Cat}$ with $\mathcal{O}(j) = \tilde{\Sigma}_j$ are symmetric monoidal categories with strict unit and strict associativity, which are also known as permutative categories (see [May74]). By analogy, having an $E_\infty$-operad in $G\text{Cat}$ allows one to define genuine permutative $G$-categories as algebras over it, and the classifying spaces of these turn out to be, after group completion, equivariant infinite loop spaces. This is carried out in the program started by Guillou and May in [GMa].

Of course, there are permutative categories, i.e. algebras over the Barratt-Eccles operad $\mathcal{O}$, which are also $G$-categories, and we reserve the name naive permutative $G$-categories for
those. The reason is that their classifying spaces are $G$-spaces, which are naive equivariant
infinite loop spaces, i.e., they have deloopings with respect to all spheres with trivial $G$
action, but not necessarily with respect to representation spheres.

**Definition 4.4.1.** The operad $\mathcal{O}_G$ in $G\text{Cat}$ defined by $\mathcal{O}_G(j) = \text{Cat}(\tilde{G}, \tilde{\Sigma}_j)$ is an $E_\infty$-$G$
operad. A genuine permutative $G$-category is defined to be an $\mathcal{O}_G$-algebra.

What are examples of genuine permutative categories? Well, if we take any naive permutative category $\mathcal{C}$, i.e., a permutative category with a $G$-action, then since it is a permutative category, it is an algebra over the Barratt-Eccles operad $\mathcal{O}$ with $\mathcal{O}(j) = \tilde{\Sigma}_j$. This means we have maps

$$\mathcal{O}(j) \times \mathcal{C}^j \rightarrow \mathcal{C}$$

compatible with the operad maps. Since $\text{Cat}(\tilde{G}, -)$ is a product preserving functor, these maps yield maps

$$\text{Cat}(\tilde{G}, \tilde{\Sigma}_j) \times \text{Cat}(\tilde{G}, \mathcal{C})^j \rightarrow \text{Cat}(\tilde{G}, \mathcal{C}),$$

and all the necessary diagrams still commute, so $\text{Cat}(\tilde{G}, \mathcal{C})$ is a genuine permutative category.

**Example 4.4.2.** Recall the definition of the category $\mathcal{G}\mathcal{L}(R)$ from Definition 2.7.16. It is a skeleton of the category of finitely generated free $R$-modules $\mathcal{F}(R)$. The category $\mathcal{G}\mathcal{L}(R)$ is permutative under direct sum of modules and block sum of matrices $\oplus : GL_n(R) \times GL_m(R) \rightarrow GL_{n+m}(R)$, since associativity and the unit are strict and commutativity holds only up to isomorphism (reordering of the basis elements by conjugation). It is a naive permutative $G$-category with trivial $G$-action on objects and entrywise $G$-action on matrices. Therefore the category $\text{Cat}(\tilde{G}, \mathcal{G}\mathcal{L}(R))$ is a genuine permutative $G$-category.

Surprisingly, the only examples of genuine permutative categories we know so far are of the form $\text{Cat}(\tilde{G}, \mathcal{C})^1$. Another open problem is to give a definition of genuine symmetric

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1. There is a slightly more general notion of categories of functors of $G$-graded categories $\tilde{G} \rightarrow \mathcal{C}$ for $G$-graded naive permutative $G$-categories $\mathcal{C}$, which yield examples of genuine permutative categories. But
monoidal $G$-categories and to understand the genuine permutative $G$-categories as a subcategory of these via some strictness condition analogously to the nonequivariant case.

The original operadic machine, which we will denote by $\mathcal{K}$, was developed in [May72]; it takes as input a permutative category and produces $\Omega$-spectra with zeroth space the group completion of the classifying space of the permutative category. An equivariant version of May's operadic infinite loop space machine is developed in [GMa]. It takes as input an $\mathcal{O}_G$-category $\mathcal{C}$, i.e., a genuine permutative $G$-category, and produces a genuine orthogonal $\Omega$-$G$-spectrum with zeroth space the equivariant group completion of $B\mathcal{C}$. We give a brief overview of the machine. As explained in [GMa], we need to use not only an $E_\infty$ operad $\mathcal{C}_G$ in $G\text{Top}$ (such as $BO_G$), but also the Steiner operads $\mathcal{K}_V$ indexed over finite dimensional subspaces of a complete $G$-universe $U$ and $\mathcal{K}_U = \text{colim} \mathcal{K}_V$. These operads are described in more detail in [GMa, Appendix]. We define the product operad

$$\mathcal{C}_V = \mathcal{C}_G \times \mathcal{K}_V.$$ 

An $\mathcal{C}_G$-space can be viewed as an $\mathcal{C}_V$-space for any $V$, and this has the advantage that $\mathcal{C}_V$ acts on $V$-fold loop spaces via its projection onto $\mathcal{K}_V$. Let $C_V$ be the monad of based $G$-spaces associated to the operads $\mathcal{C}_V$.

For a genuine permutative $G$-category $\mathcal{A}$, the orthogonal $G$-spectrum $\mathcal{K}_G(\mathcal{A})$ has spaces given by the monadic bar constructions

$$\mathcal{K}_G(\mathcal{A})(V) = B(\Sigma^V, C_V, B\mathcal{C}).$$

The structure maps for $V \subset W$ are given by

$$\Sigma^{W-V} B(\Sigma^V, C_V, X) \cong B(\Sigma^W, C_V, X) \rightarrow B(\Sigma^W, C_W, X).$$

they are not morally different from the non-graded versions, namely the ones of the form $\text{Cat}(G, \mathcal{C})$ for naive permutative categories $\mathcal{C}$.
Theorem 4.4.3 ([GMa]). For a genuine permutative $G$-category $\mathcal{C}$, the spectrum $K_G \mathcal{A}$ is a genuine $\Omega$-$G$-spectrum and there is a group completion map $B \mathcal{A} \to (K_G \mathcal{A})(0)$.

The essential properties of the machine, which we will need, are the following theorems from [GMa].

Theorem 4.4.4 ([GMa]). Let $\mathcal{A}$ and $\mathcal{B}$ be $O_G$-categories. Then the map

$$K_G(\mathcal{A} \times \mathcal{B}) \to K_G \mathcal{A} \times K_G \mathcal{B}$$

induced by the projections is a weak equivalence of $G$-spectra.

Theorem 4.4.5 ([GMa]). For $O_G$-categories $\mathcal{A}$, there is a natural weak equivalence of spectra

$$K(\mathcal{A}^G) \to (K_G \mathcal{A})^G.$$

The inclusion $\iota: O \to O_G$ induces a forgetful functor $\iota^*$ from genuine to naive permutative $G$-categories. Also, we have an inclusion $i$ of naive $G$-spectra into genuine $G$-spectra, which induces a forgetful functor $i^*$ the other way.

Theorem 4.4.6 ([GMa]). For $O_G$-categories $\mathcal{A}$, there is a natural weak equivalence of naive $G$-spectra $K \iota^* \mathcal{A} \to i^* K_G \mathcal{A}$.

By Remark 4.3.4, the $K$-theory space of a field $F$ with $G$-action is the equivariant group completion of $B\text{Cat}(\tilde{G}, \mathcal{J}\mathcal{L}(F))$. Therefore, we can define

$$K_G(F) = K_G(\text{Cat}(\tilde{G}, \mathcal{J}\mathcal{L}(F))).$$

Nonequivariantly, it is well known that using a construction of MacLane from [ML63], any symmetric monoidal category $\mathcal{C}$ can be strictified to an equivalent permutative category $\mathcal{C}^{str}$, and therefore we can apply the nonequivariant infinite loop space machine $K$ to a
symmetric monoidal category by implicitly doing this replacement first. The category \( \mathcal{C}^{str} \) has objects given by strings \((c_1, \ldots, c_n)\) of objects in \( \mathcal{C} \), and morphisms

\[(c_1, \ldots, c_n) \rightarrow (d_1, \ldots, d_m)\]
given by morphisms

\[c_1 \oplus \cdots \oplus c_n \rightarrow d_1 \oplus \cdots \oplus d_m\]
in \( \mathcal{C} \). The symmetric monoidal structure is given by concatenation and the identity is given by the empty string (\(\cdot\)).

This carries through equivariantly: if \( \mathcal{C} \) is a symmetric monoidal \( G \)-category, then \( \mathcal{C}^{str} \) is naturally also a symmetric monoidal \( G \)-category, with \( G \)-action given on objects by \( g(c_1, \ldots, c_n) = (gc_1, \ldots, gc_n) \). Since \( G \) commutes with \( \oplus \), we can define the action on morphisms by

\[g((c_1, \ldots, c_n) \xrightarrow{f} (d_1, \ldots, d_m)) = (gc_1, \ldots, gc_n) \xrightarrow{gf} (gd_1, \ldots, gd_m).\]

It is not hard to see the inverse functors in the equivalence \( \mathcal{C} \simeq \mathcal{C}^{str} \) are \( G \)-equivariant. Therefore, given a symmetric monoidal \( G \)-category \( \mathcal{C} \), the naive permutative \( G \)-category \( \mathcal{C}^{str} \) is \( G \)-equivalent to it.

Using this strictification implicitly, we can use the operadic machine on symmetric monoidal categories. We give the following definition for all \( G \)-rings:

\[\mathbf{K}_G(R) = \mathbb{K}_G(\text{Cat}((\tilde{G}, \mathcal{P}(R))),\)

with the understanding that we have replaced the symmetric monoidal \( G \)-category \( \mathcal{P}(R) \) with an equivalent naive permutative \( G \)-category. Alternatively, we can use the equivariant Segal machine, which we address in the next section.
4.4.2 Equivariant Segalic machine

Segal developed an alternative delooping machine to the operadic machine in the celebrated paper [Seg74], which we will denote as \( S \). The input is a \( \Gamma \)-space, which is just a functor

\[
X : \mathcal{F} \to \text{Top}, \quad n \mapsto X_n,
\]

where \( \mathcal{F} \) is a skeleton of the category of based finite sets\(^2\). A \( \Gamma \)-space is *special* if the map \( \delta : X_n \to X_1^n \), induced by the projections \( \delta_i : n \to 1 \), is an equivalence. From a \( \Gamma \)-space, Segal produces a spectrum, and he shows that for a special \( \Gamma \)-space, the spectrum is \( \Omega \), with zeroth space the group completion of \( X_1 \).

One can start with a \( \Gamma \)-category instead, i.e., a functor \( \mathcal{F} \to \text{Cat} \) and define it to be special if the \( \Gamma \)-space obtained by applying the classifying space functor levelwise is a special \( \Gamma \)-space. Segal gives a construction of a special \( \Gamma \)-space \( X \) from a symmetric monoidal category \( \mathcal{C} \), with \( X_1 \simeq \mathcal{C} \). Therefore, \( S(\mathcal{C}) \), the spectrum obtained from the special \( \Gamma \)-space associated to the symmetric monoidal category \( \mathcal{C} \), is \( \Omega \), with zeroth space the group completion of \( B\mathcal{C} \).

Shimakawa has generalized Segal’s machine in [Shi89] to produce an orthogonal genuine \( \Omega-G \)-spectrum starting from a special \( \Gamma_G \)-space. A \( \Gamma_G \)-space is a functor

\[
X : \mathcal{F}_G \to \text{Top}_G, \quad A \mapsto X(A),
\]

where \( \mathcal{F}_G \) is the category of finite \( G \)-sets and \( \text{Top}_G \) is the category of \( G \)-spaces and nonequivariant based maps; \( G \) acts by conjugation on morphism sets. For any \( A \in \mathcal{F}_G \), we have a projection \( \delta_A : A \to 1 \), which sends all the nonbasepoint elements of \( A \) to 1 and the basepoint to 0. A \( \Gamma_G \)-space is special if the map \( \delta_A : X(A) \to \text{Map}(A, X_1) \) induced by these projections is a \( G \)-equivalence for all \( A \in \mathcal{F}_G \). We note that this map turns out to be a \( G \)-map even

---

\(^2\) The opposite of Segal’s original category \( \Gamma \) turns out to be just \( \mathcal{F} \).
though the individual maps \( \delta_a \) are generally not \( G \)-maps.

Given a \( \Gamma_G \)-space \( X \), Shimakawa constructs a spectrum \( S_G X \) with \( V \)th space given by the two-sided bar construction \( B((S^V)\bullet, \mathcal{F}_G, X) \), where \( (S^V)\bullet \) is the contravariant functor \( \mathcal{F}_G \to \text{Top}_G \) defined on objects as \( A \mapsto \text{Map}(A, S^V) \). It is not hard to see that there are structure maps

\[
S^W \wedge B((S^V)\bullet, \mathcal{F}_G, X) \to B((S^{V\oplus W})\bullet, \mathcal{F}_G, X).
\]

The following is the main theorem in [Shi89].

**Theorem 4.4.7** ([Shi89]). For a special \( \Gamma_G \)-space \( X \), the spectrum \( S_G X \) is a genuine \( \Omega_G \)-spectrum, for which \( X_1 \simeq (S_G X)(0) \) if and only if \( X_1 \) is group like.

Essential to our applications is that in general there is a group completion map \( X_1 \to (S_G X)(0) \), which Shimakawa does not prove, but we fill this gap in [MMO].

A \( \Gamma_G \)-category is a functor \( \mathcal{F}_G \to \text{Cat}_G \), where \( \text{Cat}_G \) is the category of \( G \)-categories and nonequivariant functors. It is special if the \( \Gamma_G \)-space obtained by applying the classifying space functor levelwise is special. Shimakawa generalizes Segal’s combinatorial way of constructing a \( \Gamma \)-category from a symmetric monoidal category to constructing a \( \Gamma_G \)-category from a symmetric monoidal \( G \)-category \( \mathcal{C} \). This \( \Gamma_G \)-category is not necessarily special, but Shimakawa shows that replacing \( \mathcal{C} \) by the symmetric monoidal \( G \)-category \( \text{Cat}(\widetilde{G}, \mathcal{C}) \) does yield a special \( \Gamma_G \)-category, and therefore, \( S_G(\text{Cat}(\widetilde{G}, \mathcal{C})) \), the machine applied to the special \( \Gamma_G \)-category obtained from the symmetric monoidal \( G \)-category \( \text{Cat}(\widetilde{G}, \mathcal{C}) \), is a genuine orthogonal \( \Omega_G \)-spectrum with zeroth space the group completion of \( BC\text{at}(\widetilde{G}, \mathcal{C}) \).

### 4.4.3 Agreement of equivariant infinite loop space machines

The question of whether the May and Segal approaches produce the same genuine \( G \)-spectrum for a symmetric monoidal \( G \)-category is of general interest because each machine has its own advantages, and it is therefore important for equivariant algebraic \( K \)-theory if
one wants to use either of the two machines interchangeably.

Nonequivariantly, the celebrated theorem of May and Thomason from [MT78] states the uniqueness of infinite loop space machines, thus showing that the May and Segal delooping machines produce the same algebraic $K$-theory spectrum when fed equivalent symmetric monoidal categories. The proof, however, does not generalize equivariantly; it relies on an iterative definition of the Segal machine, which cannot be adapted since a genuine $G$-spectrum is indexed on representations and not on integers. In [MMO], we prove via a surprising chain of equivalences that the two machines do in fact agree equivariantly.

**Theorem 4.4.8.** [MMO] For a symmetric monoidal $G$-category $\mathcal{C}$ we have an equivalence of orthogonal $G$-spectra $\mathbb{K}_G(\text{Cat}(\widetilde{G}, \mathcal{C})) \simeq S_G(\text{Cat}(\widetilde{G}, \mathcal{C}))$.

Therefore, from now on, when we write $\mathbb{K}_G(R)$ we can mean either of the equivalent $\Omega$-$G$-spectra $\mathbb{K}_G(\text{Cat}(\widetilde{G}, \mathcal{P}(R)))$ or $S_G(\text{Cat}(\widetilde{G}, \mathcal{P}(R)))$.

An immediate consequence of this definition and Proposition 2.7.14 is the following theorem, which says that we recover the classical nonequivariant $K$-theory of twisted group rings as the fixed points of our construction.

**Theorem 4.4.9.** If $H \subseteq G$ and $|H|^{-1} \in R$, there is an equivalence of spectra

\[
\mathbb{K}_G(R)^H \simeq K(R_H[H]).
\]

4.4.4 Functoriality

**Theorem 4.4.10.** The assignment $R \mapsto \mathbb{K}_G(R)$ is a functor from the category of $G$-rings and $G$-maps to genuine orthogonal $G$-spectra.

**Proof.** We have already constructed a genuine $G$-spectrum $\mathbb{K}_G(R)$ from a $G$-ring $R$. We need to show that having a map of $G$-rings $R \to S$ yields a map of genuine $G$-spectra.

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Suppose $f: R \to S$ is a $G$-map of $G$-rings, and consider the functor $\mathcal{P}(R) \to \mathcal{P}(S)$
defined as $M \mapsto M \otimes_R S$. Note that certainly $gM \otimes_R S \neq g(M \otimes_R S)$ since the scalar
multiplication is different in the two modules; however we go on to define an isomorphism

$$gM \otimes_R S \cong g(M \otimes_R S),$$

which says that the functor $- \otimes_R S$ is pseudo equivariant.

Recall that in $gM$, the scalar multiplication is defined as $r \cdot g m = r^g m$, where $(-)^g$ denote
the action of $g$ on $R$. Define

$$gM \otimes_R S \to g(M \otimes_R S), \quad m \otimes s \mapsto m \otimes s^g.$$

First of all, we use the assumption that $f$ is a $G$-map to show that this assignment is
well-defined. Note that for $t \in R$, we have the following identification in $gM \otimes_R S$:

$$m \otimes ts \sim f(t) \cdot g m \otimes s = f(t)^g m \otimes s,$$

Now

$$m \otimes ts \mapsto m \otimes (ts)^g = m \otimes t^g s^g \sim f(t^g) m \otimes s^g,$$

and

$$f(t)^g m \otimes s \mapsto f(t)^g m \otimes s^g,$$

but these are equal since $f(t^g) = f(t)^g$.

We check next that the assignment is $S$-linear: for $t \in S$,

$$t(m \otimes s) = m \otimes ts \mapsto m \otimes (ts)^g = m \otimes t^g s^g = t \cdot g (m \otimes s^g).$$
Similarly, we can check that the inverse map

\[ g(M \otimes_R S) \rightarrow gM \otimes_R S, \quad m \otimes s \mapsto m \otimes s^{(g^{-1})} \]

is well-defined and $S$-linear, so that we have the claimed isomorphism.

Thus the functor $- \otimes_R S$ is pseudo equivariant, and by Proposition 2.4.4 it yields a $G$-map

\[ \text{Cat}(\tilde{G}, \mathcal{P}(R)) \rightarrow \text{Cat}(\tilde{G}, \mathcal{P}(S)), \]

which in turn gives a map of genuine $G$-spectra

\[ K_G(R) \rightarrow K_G(S) \]

by the functoriality of the equivariant infinite loop space machine.

\[ \square \]

### 4.5 Equivariant $K$-theory of Galois extensions

The algebraic $K$-theory of Galois extensions behaves particularly nicely as a result of faithfully flat descent and the fact that for $G$-Galois extensions the category of descent data has an interpretation in terms of modules with semilinear $G$-action. We briefly review faithfully flat descent and Galois extensions, following the detailed expositions given in [Cha13], [CRi] and [Vis08]. For all the details, we refer the reader to these excellent sources.

#### 4.5.1 Faithfully flat descent

Let $R \rightarrow S$ be a faithfully flat extension of rings. Given an $R$-module $M$, we naturally get an $S$-module $M \otimes_R S$ by extension of scalars. This is called ascent. We can ask the reverse question: given an $S$-module, when is it obtained from an $R$-module by extension of scalars? This is called descent. For faithfully flat extensions of rings, we can give an answer to this
question.

If $N$ is an $S$-module, there are two $S \otimes_R S$-module structures on $N \otimes_R S$ given by

$$(a \otimes b)(n \otimes s) = an \otimes bs \quad \text{and} \quad (a \otimes b)(n \otimes s) = bn \otimes as$$

for $n \in N$ and $a,b,s \in S$.

**Definition 4.5.1.** Let $R \to S$ be a faithfully flat ring extension. The *category of $S$-modules with descent data* is defined as follows. The objects are pairs $(N, \phi)$, where $N$ is an $S$-module and $\phi$ is an isomorphism $\phi : N \otimes_R S \cong N \otimes_R S$ between the two module structures on $N \otimes_R S$, making the diagram commute, where $\phi_i$ is defined in terms of $\phi$ in the following way: if $\phi(n \otimes s) = \sum n_i \otimes s_i$, then for $n \in N$ and $s,t \in S$,

\[
\begin{align*}
\phi_0(n \otimes t \otimes s) &= \sum n_i \otimes t \otimes s_i, \\
\phi_1(n \otimes t \otimes s) &= \sum n_i \otimes s_i \otimes t, \\
\phi_2(n \otimes s \otimes t) &= \sum n_i \otimes s_i \otimes t.
\end{align*}
\]

A morphism $(N, \phi) \to (N', \phi')$ is given by an $S$-module isomorphism $f : N \to N'$ that makes the diagram commute.

We define a functor from the category $\text{Mod}(R)$ of $R$-modules to the category of $S$-modules
with descent data for $R \to S$ by

$$M \mapsto (M \otimes_R S, \phi),$$

where $\phi : M \otimes_R S \otimes_R S \xrightarrow{\cong} M \otimes_R S \otimes_R S$ is given by $m \otimes s \otimes t \mapsto m \otimes t \otimes s$. It is easy to see that $\phi_0 \phi_2 = \phi_1$, so that $(M \otimes_R S, \phi)$ is indeed an object of the category of descent data.

We state the main descent theorem, a great proof of which is given in [Vis08]:

**Theorem 4.5.2** (Faithfully flat descent). The above functor $M \mapsto (M \otimes_R S, \phi)$ from $\text{Mod}(R)$ to the category of $S$-modules with decent data for the faithfully flat extension $R \to S$, is an equivalence of categories.

Let us just indicate that the key to the above theorem is that faithful flatness allows the generalization of the following observation for $M \otimes_R S$ to general modules with descent data: An element $m \otimes s$ of $M \otimes_R S$ is in $M$ if $m \otimes s \otimes 1$ and $m \otimes 1 \otimes s$ are identified in $M \otimes_R S \otimes_R S$, i.e., we have an exact sequence:

$$0 \to M \to M \otimes_R S \xrightarrow{p^M_1 - p^M_2} M \otimes_R S \otimes_R S,$$

where $p^M_1(m \otimes s) = m \otimes s \otimes 1$ and $p^M_2(m \otimes s) = m \otimes 1 \otimes s$. For example, for the faithful flat extension $\mathbb{R} \to \mathbb{C}$, and $M = \mathbb{R}$ we have an exact sequence

$$\mathbb{R} \to \mathbb{C} \xrightarrow{x \otimes 1 - 1 \otimes x} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C},$$

since the elements in $\mathbb{C}$ which commute past the tensor are precisely the elements in $\mathbb{R}$.

### 4.5.2 Galois extensions

An early treatment of Galois extensions of rings is given in [CHR65]. We will follow the lecture notes [Cha13] to give a definition of Galois extensions of rings and state examples of
such. For a ring extension $R \to S$, let $\text{Aut}_R(S)$ be the group of automorphisms of $S$ fixing $R$.

**Definition 4.5.3.** Let $R \to S$ be a faithfully flat ring extension and suppose that $G$ is a finite subgroup of $\text{Aut}_R(S)$. The extension $R \to S$ is Galois with Galois group $G$ if the map

$$\gamma: S \otimes_R S \to \prod_G S, \quad a \otimes b \mapsto ((g \cdot a)b)_{g \in G}. \quad (4.5)$$

is an $S$-algebra isomorphism.

**Proposition 4.5.4.** If $R \to S$ is a Galois extension with Galois group $G$, then $R = S^G$.

**Proof.** Since $R \to S$ is faithfully flat, the sequence

$$0 \to R \to S \xrightarrow{s \otimes 1 - 1 \otimes s} S \otimes_R S$$

is exact, so

$$R = \{s \in S \mid s \otimes 1 = 1 \otimes s\}.$$

Since $\gamma$ is an isomorphism, $s \otimes 1 = 1 \otimes s$ if and only if $\gamma(s \otimes 1) = \gamma(1 \otimes s)$, thus

$$R = \{s \in S \mid g \cdot s = s \quad \forall g \in G\}.$$

The following examples from [Cha13] are important because we can apply the theory of Galois extensions to obtain results about their equivariant $K$-theory.

**Example 4.5.5.** As one might expect, any finite field Galois extension satisfies the above definition.

**Example 4.5.6.** For any ring $R$, the diagonal map $\Delta: R \to R \times R$ is a Galois extension with group $\mathbb{Z}/2\mathbb{Z}$, where the nontrivial element acts on $R \times R$ by interchanging the factors. The
isomorphism \( \gamma : S \otimes_R S \to S \times S \) is given by

\[
(a, b) \otimes (c, d) \mapsto (ac, bd, bc, ad)
\]

and has inverse

\[
(a, b, c, d) \mapsto (1, 0) \otimes (a, b) + (0, 1) \otimes (c, d).
\]

Recall that if \( S \) is a \( G \)-ring, an \( S \)-module with semilinear \( G \)-action is an \( S \)-module \( M \) with an action

\[
G \times M \to M,
\]

which is semilinear in the sense that

\[
g \cdot (sm) = s^g (g \cdot m).
\]

The category of modules with semilinear \( G \)-action has morphisms the \( G \)-equivariant maps. Now the wonderful fact about Galois extensions is the following theorem, for whose proof we once again refer to [Cha13].

**Theorem 4.5.7.** Suppose \( R \to S \) is a Galois ring extension with Galois group \( G \). Then the category of \( S \)-modules with descent data is equivalent to the category of \( S \)-modules with semilinear \( G \)-action.

**Corollary 4.5.8.** Suppose \( R \to S \) is a Galois ring extension with Galois group \( G \). Then the category of \( R \)-modules is equivalent to the category of \( S \)-modules with semilinear \( G \)-action.

**Proof.** This follows immediately from Theorem 4.5.2 combined with Theorem 4.5.7. \( \square \)
4.5.3 Equivariant K-theory of Galois extensions

In the proof of Proposition 2.7.12 we showed that for a $G$-ring $S$, the category of $S$-modules with semilinear $G$-action is equivalent to the homotopy fixed point category $\text{Cat}(\tilde{G}, \text{Mod}(S))^G$. We reinterpret Corollary 4.5.8:

**Proposition 4.5.9.** Suppose $R \to S$ is a Galois ring extension with Galois group $G$. Then there is an equivalence of categories

$$\text{Mod}(R) \simeq \text{Cat}(\tilde{G}, \text{Mod}(S))^G.$$ 

Suppose $R \to S$ is faithfully flat. Then an $R$-module $M$ is finitely generated projective if and only the $S$-module $M \otimes_R S$ is finitely generated projective (see [Cha13, Prop. 2.12.]). Therefore, the equivalence of categories in Theorem 4.5.2 restricts to finitely generated projective modules. Similarly, the equivalence in Theorem 4.5.7 restricts to the corresponding categories of finitely generated projective modules, and we obtain the analogue of Corollary 4.5.8 and its above reinterpretation:

**Proposition 4.5.10.** Suppose $R \to S$ is a Galois ring extension with Galois group $G$. Then there is an equivalence of categories

$$\mathcal{P}(R) \simeq \text{Cat}(\tilde{G}, \mathcal{P}(S))^G.$$ 

This leads to the following theorem, which says that for a $G$-Galois extension $R \to S$, the $G$-fixed points of the $G$-equivariant $K$-theory of $S$ is the same as nonequivariant $K$-theory of the fixed ring $S^G = R$.

**Theorem 4.5.11.** Let $R \to S$ be a Galois extension of rings with Galois group $G$. Then
there is an equivalence of orthogonal spectra

\[ K_G(S)^G \simeq K(R). \]

**Proof.** By Proposition 4.5.10, we have an equivalence of categories

\[ \mathcal{P}(R) \simeq \text{Cat}(\tilde{G}, \mathcal{P}(S))^G. \]

By Theorem 4.4.5, we have

\[ K_G(\text{Cat}(\tilde{G}, \mathcal{P}(S))^G)^G \simeq K(\text{Cat}(\tilde{G}, \mathcal{P}(S))^G)^G. \]

Therefore,

\[ K_G(S)^G \simeq K(R). \]

\[ \square \]

**Example 4.5.12.** For any \( G \)-Galois extension \( E/F \),

\[ K_G(E)^G \simeq K(F). \]

In particular, this recovers \( K(\mathbb{Q}) \) as the fixed point spectrum of the genuine equivariant \( K \)-theory spectrum of any Galois extension of \( \mathbb{Q} \).

**Example 4.5.13.** For any ring \( R \), from the Galois extension \( R \to R \times R \) described in Example 4.5.6,

\[ K_{\mathbb{Z}/2\mathbb{Z}}(R \times R)^{\mathbb{Z}/2\mathbb{Z}} \simeq K(R). \]
4.5.4 Relation to Hilbert 90

In [Del77] faithfully flat descent is used to prove Hilbert’s theorem 90, which we recall here. Suppose $E/F$ is a finite field Galois extension.

**Theorem 4.5.14** (Hilbert 90). The cohomology group $H^1(G, E^\times)$ is trivial.

In the same spirit, we use Corollary 4.5.8 to give a proof of the generalization of Hilbert’s theorem 90, which is due to Serre:

**Theorem 4.5.15** (Serre). The nonabelian cohomology $H^1(G, GL_n(E)_G)$ is trivial.

**Proof.** Faithfully flat descent gives an equivalence of categories between the category of $F$-modules and the category of descent data, i.e., the category of $E$-modules with semilinear $G$-action. From the proof of Proposition 2.7.12, the latter is just $\text{Cat}(\tilde{G}, \text{iso Mod}(E))^G$. Note that $\coprod_n GL_n(E)$ is a skeleton of $\text{iso Mod}(E)$, and by Proposition 2.3.12 we have a weak equivalence

$$\text{Cat}(\tilde{G}, \text{iso Mod}(E))^G \simeq \text{Cat}(\tilde{G}, \coprod_n GL_n(E))^G.$$  

Using that $\coprod_n GL_n(F)$ is a skeleton for $\text{iso Mod}(F)$, by the description of the fixed points of $\text{Cat}(\tilde{G}, GL_n(E))$ given in Theorem 2.5.4, we have

$$\coprod_n GL_n(F) \simeq \coprod_n \text{Cat}_\times(G, GL_n(E)).$$

Therefore, for any summand in the coproduct, there is only one isomorphism class of objects. Now we note that the isomorphism set of objects in $\text{Cat}_\times(G, GL_n(E))$ is precisely the first nonabelian cohomology set $H^1(G, GL_n(E)_G)$.

This gives the following result, which we could have used directly to conclude that for a finite field Galois extension $E/F$ with Galois group $G$ we have an equivalence $K_G(E)^G \simeq K(F)$.  

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Proposition 4.5.16. There is a symmetric monoidal weak $G$-equivalence $\iota: \mathcal{G}(E) \to \text{Cat}(\tilde{G}, \mathcal{G}(E))$.

Proof. By Proposition 2.5.8, we have that the map

$$\iota: \mathcal{G}(E)^H \to \text{Cat}(\tilde{G}, \mathcal{G}(E))^H$$

is an equivalence precisely when $H^1(G, GL_n(E)_G)$ is trivial, and this is true in this case by Theorem 4.5.15. Now also note that

$$\mathcal{G}(E)^H = \mathcal{G}(E^H).$$

This completes the proof. \qed

4.5.5 Remarks on profinite Galois extensions

Let $k$ be a field with separable closure $\bar{k}$ and absolute Galois group $\text{Gal}(\bar{k}/k)$. Then

$$\text{Gal}(\bar{k}/k) = \text{colim} \text{Gal}(L/k),$$

where the colimit runs over all finite Galois extensions $L/k$. Note that

$$\text{Gal}(L/k) = \text{Gal}(\bar{k}/k)/\text{Gal}(\bar{k}/L).$$

The absolute Galois group is a profinite group with the profinite group topology, and it acts continuously through group homomorphisms on the discrete group $\bar{k}^\times$, or more generally, on $GL_n(\bar{k})$. 

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There are inclusions

\[ H^1(\text{Gal}(L/k), GL_n(L)) \hookrightarrow H^1(\text{Gal}(\bar{k}/k), GL_n(\bar{k})) \]

coming from the inclusions of cocycles, which by definition need to be continuous crossed homomorphisms. We get

\[ H^1(\text{Gal}(\bar{k}/k), GL_n(\bar{k}) = \text{colim} \ H^1(\text{Gal}(L/k), GL_n(\bar{k})), \]

where the colimit runs over all finite Galois extensions.

Thus we get an equivalence analogous to the one in proposition Proposition 4.5.16:

\[ \mathcal{G} \mathcal{L}(k) \xrightarrow{\sim} \text{Cat}(\text{Gal}(\bar{k}/k), \mathcal{G} \mathcal{L}(\bar{k}))^{\text{Gal}(\bar{k}/k)}. \]

However, the equivariant infinite loop space machines, as currently developed, do not apply to profinite groups, so we cannot pass from this statement to a spectrum level statement. We do hope, though, that in future work we will be able to generalize the delooping machines to profinite groups.

\subsection*{4.5.6 Quillen-Lichtenbaum formulation}

Let $E/F$ be a field Galois extension. It has been long suspected that the $K$-theory of a field $F$ should depend on $K$-theory of $E$ and the action of the Galois group $G = \text{Gal}(E/F)$ on $E$. Since $G$ acts on $E$, it acts by functoriality on the spectrum $KE$, so $KE$ is a naive $G$-spectrum. The fixed points of this naive $G$-spectrum are easily seen to be $(KE)^G \simeq KF$. 

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The initial Quillen-Lichtenbaum conjecture was that the map of spectra

\[ \mathbf{K} \mathcal{F} \simeq \mathbf{K} \mathcal{E}^G \rightarrow \mathbf{K} \mathcal{E}^{hG} \]

is an equivalence, where \( \mathbf{K} \mathcal{E}^{hG} \) denotes the homotopy fixed points of the naive spectrum \( \mathbf{K} \mathcal{E} \), which was defined in section 2.1.1. However, low dimensional examples disprove this conjecture as stated even after \( p \)-adic completion for a prime \( p \). Thomason showed that this map only becomes an equivalence after reducing mod a prime power and inverting the Bott element.

The Quillen-Lichtenbaum conjecture was refined to the statement that if \( \bar{\mathcal{F}} \) is the algebraic closure of \( \mathcal{F} \) and \( \mathcal{G} \) is the absolute Galois group, then the map

\[ \mathbf{K} \mathcal{F}^\wedge_p \simeq (\mathbf{K} \bar{\mathcal{F}}^\wedge)^G \rightarrow \mathbf{K} \bar{\mathcal{F}}^{hG} \]

induces an isomorphism on \( \pi_i \) for \( i \) greater than the cohomological dimension of \( \mathcal{G} \). This means that the homotopy limit spectral sequence

\[ E_2^{p,q} = H^p(\mathcal{G}, \pi_q(\mathbf{K} \bar{\mathcal{F}}^\wedge)) \Rightarrow \pi_{q-p}\mathbf{K} \mathcal{F}^\wedge_p, \]

described in Section 2.1.1, converges in high enough degrees. This conjecture has been settled by Rost and Voevodsky.

We can restate the Quillen-Lichtenbaum conjecture for a finite Galois extension in the setting of equivariant algebraic K-theory. We review the definition of homotopy fixed points of a fibrant genuine orthogonal \( \mathcal{G} \)-spectrum \( \mathcal{X} \). Note that a fibrant spectrum is an \( \Omega-\mathcal{G} \)-spectrum.

**Definition 4.5.17.** Let \( \mathcal{X} \) be a fibrant genuine orthogonal \( \mathcal{G} \)-spectrum. Then \( \mathcal{X}^{h\mathcal{G}} \) is defined
to be the fixed point spectrum

\[ \text{Map}_*(EG_+, X)^G = (i^* \text{Map}_*(EG_+, X))^G \simeq \text{Map}_*(EG_+, i^*X)^G, \]

where \( i^* \) is the forgetful functor from genuine to naive \( G \)-spectra.

Just as for \( G \)-spaces and naive \( G \)-spectra, we have a natural map

\[ X^G \to X^{hG}, \]

induced by the projection \( EG_+ \to S^0 \).

Now let \( E/F \) be a finite Galois extension with Galois group \( G \). From Example 4.5.12, we have that \( K_F \simeq K_G(E)^G \), and we get a map

\[ K_F \simeq K_G(E)^G \to K_G(E)^{hG}. \]

We show in the next proposition that \( K_G(E)^{hG} \simeq K(E)^{hG} \), where on the left hand side we are taking homotopy fixed points of a genuine \( G \)-spectrum, and on the right hand side we are taking homotopy fixed points of a naive \( G \)-spectrum. Thus we recover the initial form of the Quillen-Lichtenbaum conjecture as a statement about genuine \( G \)-spectra.

**Proposition 4.5.18.** For a Galois extension \( E/F \) with Galois group \( G \), the homotopy fixed points of the naive \( G \)-spectrum \( KE \) and the homotopy fixed points of the genuine \( G \)-spectrum \( K_G(E) \) are equivalent.

**Proof.** We will show that we have an equivalence of naive \( G \)-spectra \( i^*K_G(E) \simeq KE \), which will imply the result. Recall that \( KE \) is defined as the \( K \) theory of the naive permutative \( G \)-category \( \mathcal{G}\mathcal{L}(E) \), while \( K_G(E) \) is the equivariant algebraic \( K \)-theory of the genuine
permutative $G$-category $\text{Cat}(\tilde{G}, \mathcal{L}(E))$. We have a map

$$K(\mathcal{L}(E)) \to K(i^*(\text{Cat}(\tilde{G}, \mathcal{L}(E)))) \xrightarrow{\sim} i^*K_G(\text{Cat}(\tilde{G}, \mathcal{L}(E))),$$

where the second map is shown to be an equivalence in [GMa].

By Proposition 4.5.16, there is a symmetric monoidal weak $G$-equivalence

$$\mathcal{L}(E) \simeq i^*(\text{Cat}(\tilde{G}, \mathcal{L}(E))),$$

so the first map is also an equivalence.

\[\square\]

### 4.6 Carlsson’s program

There has been a long standing program initiated and lead by G. Carlsson of studying the $K$-theory of fields motivated by the concept of descent and the Quillen-Lichtenbaum conjecture. Carlsson conjectured that there is a spectrum which depends only on the absolute Galois group of a field $F$ and whose homotopy groups are the $p$-completed homotopy groups of $K_F$. Whereas the above mentioned spectral sequence figuring in the theorem of Rost and Voevodsky yields information about the homotopy groups of the spectrum $K_F$, Carlsson’s program would give an actual homotopy theoretic model for it, which could be used in applications where knowledge of just the homotopy groups might be too weak. Carlsson’s conjecture has been studied and proved for a lot of cases in [Car11, Lyo07] and more recently in [Car, CR].
4.6.1 Carlsson’s conjecture

Suppose that $E/F$ is a Galois extension with Galois group $G$. We can consider the assembly map

$$\text{Rep}_F[G] \xrightarrow{E \otimes_F -} V^G(E), \quad (4.6)$$

given by extension of scalars from the category of continuous finite dimensional $G$-representations in $F$, which is denoted by $\text{Rep}_F[G]$ in [Car11], to the category of finite dimensional $E$-vector spaces with semilinear $G$-action, or equivalently, the category of $E$-vector spaces with descent data for the faithfully flat extension $E/F$, denoted by $V^G(E)$ in [Car11]. By Corollary 4.5.8, the target category $V^G(E)$ is equivalent to the category $\text{Vect}(F)$ of finite dimensional $F$-vector spaces.

Now consider the Galois extension $\bar{F}/F$ with absolute Galois group $G$ and suppose $k \hookrightarrow F$ is an algebraically closed subfield. We can precompose the maps above with $\text{Rep}_G[k] \to \text{Rep}_G[F]$, to get an assembly map

$$\text{Rep}_k[G] \to \text{Rep}_F[G] \xrightarrow{\bar{F} \otimes_F -} V^G(E) \simeq \text{Vect}(F). \quad (4.7)$$

Since all these categories are nonequivariant symmetric monoidal categories, their $K$-theory spectra are defined by using standard nonequivariant infinite loop space machines such as the May [May72] or the Segal machine [Seg74].

**Conjecture 4.6.1** (Carlsson). Consider the Galois extension $\bar{F}/F$ with absolute Galois group $G$ and suppose $k \hookrightarrow F$ is an algebraically closed subfield. Then the assembly map

$$\text{Rep}_k[G] \to V^G(\bar{F}) \simeq \text{Vect}(F)$$
induces an equivalence on $K$-theory after derived completion, i.e.,

$$K(\text{Rep}_k[G])_{\alpha_p}^\wedge \to K(F)_{\alpha_p}^\wedge \simeq K(F)_{p}^\wedge.$$

**Note 4.6.2.** Carlsson defines the derived completion of a ring spectrum in [Car08] and shows that for $KF$ the derived completion agrees with the Bousfield-Kan $p$ completion, which accounts for the identification made above on the right hand side.

This conjecture has recently been proved by G. Carlsson for pro-$l$ absolute Galois groups in [Car] and [CR]. Attacking the problem from the perspective of full-fledged equivariant spectra might eventually shed some light on the general case. C. Barwick has a sketch of such a proof from an $\infty$-categorical point of view.

### 4.6.2 Interpretation of Carlsson’s map as fixed points of an equivariant map

We interpret the map in the conjecture from an equivariant perspective. More precisely, we show how to construct an equivariant map, which restricts to Carlsson’s assembly map on fixed points. On the categorical level this makes sense for the separable Galois extension of a field and profinite Galois group; however, we only know how to obtain a map of genuine $G$-spectra for a finite Galois extension at the moment, because of the limitations of the equivariant infinite loop space machines.

Suppose $E/F$ is a Galois extension with Galois group $G$, so that $G$ acts on $E$ by the Galois action and $G$ acts trivially on the fixed field $F = E^G$.

**Note 4.6.3.** From Proposition 2.7.12, the category $V^G(E)$ of $E$-vector spaces with semilinear $G$-action is isomorphic to $\text{Cat}(\tilde{G}, \text{Vect}(F))^G$.

**Note 4.6.4.** As we have remarked before, if $G$ is acting trivially on $F$, $\text{Cat}(\tilde{G}, \text{Vect}(F))^G \cong \text{Cat}(\tilde{G}, \text{Vect}(F))^G \cong$
Cat\((G, \text{Vect}(F))\), the category of \(F\)-vector spaces with \(G\)-linear action, i.e., the category of continuous \(G\)-representations over \(F\). Thus \(\text{Rep}_F[G] \cong \text{Cat} (\tilde{G}, \text{Vect}(F))^G\).

Therefore, the assembly map (4.6), translates to a map

\[
\text{Cat} (\tilde{G}, \text{Vect}(F))^G \longrightarrow \text{Cat} (\tilde{G}, \text{Vect}(E))^G.
\]

Suppose that \(k \hookrightarrow F\) is an algebraically closed subfield. The inclusion gives a functor \(\text{Vect}(F) \longrightarrow \text{Vect}(k)\), which is clearly equivariant since the action of \(G\) on both categories is trivial, so this induces an equivariant functor

\[
\text{Cat} (\tilde{G}, \text{Vect}(k)) \longrightarrow \text{Cat} (\tilde{G}, \text{Vect}(F)), \quad (4.8)
\]

which induces maps on fixed points

\[
\text{Rep}_k[G] = \text{Cat} (\tilde{G}, \text{Vect}(k))^G \longrightarrow \text{Cat} (\tilde{G}, \text{Vect}(F))^G = \text{Rep}_F[G].
\]

Therefore, the assembly map (4.7), translates into a map

\[
\text{Cat} (\tilde{G}, \text{Vect}(k))^G \longrightarrow \text{Cat} (\tilde{G}, \text{Vect}(F))^G.
\]

It remains to show that we have a \(G\)-map

\[
\text{Cat} (\tilde{G}, \text{Vect}(k)) \longrightarrow \text{Cat} (\tilde{G}, \text{Vect}(F)),
\]

which induces this map on fixed points. Since the functor in (4.8) is equivariant, it suffices to show that we have an equivariant map

\[
\text{Cat} (\tilde{G}, \text{Vect}(F)) \longrightarrow \text{Cat} (\tilde{G}, \text{Vect}(E)),
\]

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which restricts to the assembly map (4.6) on fixed points.

Recall that the action of $G$ on $\text{Vect}(E)$ is given by twisting the scalar multiplication. Note that the extension of scalars map $\text{Vect}(F) \xrightarrow{\otimes_F} \text{Vect}(E)$ is not a $G$-map. The action of $G$ on $\text{Vect}(F)$ is trivial; however, for $V \in \text{Vect}(F)$, the object $E \otimes_F V$ is not $G$-fixed:

$$E \otimes_F V \neq g(E \otimes_F V)$$

since they have different scalar multiplication.

However, we have shown that for free $R$-modules $M$ over a $G$-ring $R$, we have that $gM \cong M$. Thus in this case,

$$E \otimes_F V \cong g(E \otimes_F V)$$

and the extension of scalars functor is pseudo equivariant. By Proposition 2.4.4 this induces an equivariant map $\text{Cat}(\tilde{G}, \text{Vect}(k)) \rightarrow \text{Cat}(\tilde{G}, \text{Vect}(F))$.

Therefore, if $E/F$ is a finite Galois extension, by applying the equivariant infinite loop space machine, we get a $G$-map of genuine $G$-spectra

$$\mathbf{K}_G(F) \rightarrow \mathbf{K}_G(E),$$

which restricts to Carlsson’s map on fixed point spectra.

### 4.7 Equivariant algebraic $K$-theory of topological rings

We first note that in this section any time we refer to topological $K$-theory, we mean the connective version. We describe in this section how our construction of equivariant algebraic $K$-theory recovers the well-known equivariant topological real and complex $K$-theories $\mathbf{K}U_G$ and $\mathbf{K}O_G$, as defined in [Seg68] and Atiyah’s Real $K$-theory $\mathbf{K}R$, as defined in [Ati66]. We denote by $\mathbf{k}U_G$, $\mathbf{k}O_G$ and $\mathbf{k}R$ their connective covers. Whereas the first two are well studied,
the latter is not so well-known. A construction of the connective version of $KR$ is given, for example, in [Dug05].

### 4.7.1 Equivariant complex and topological $K$-theory

We observe that the representing spaces for complex and real topological $K$-theory, namely $BU \times \mathbb{Z}$ and $BO \times \mathbb{Z}$, are the group completions of the topological monoids $\coprod BU_n$ and $\coprod BO_n$, respectively. Note that as topological spaces

$$GL_n(\mathbb{C}) \simeq U_n \text{ and } GL_n(\mathbb{R}) \simeq O_n,$$

(4.9)

and it is not hard to see that

$$BGL_n(\mathbb{C}) \simeq BU_n \text{ and } BGL_n(\mathbb{R}) \simeq BO_n,$$

(4.10)

thus

$$K(\mathbb{C}) \simeq ku \text{ and } K(\mathbb{R}) \simeq ko,$$

since they are connective $\Omega$-spectra with zeroth spaces $BU \times \mathbb{Z}$ and $BO \times \mathbb{Z}$, respectively.

The same story carries through equivariantly for any finite group $G$, when the group action is trivial. Our definition of the algebraic $K$-theory spectrum applies to topological rings, since the equivariant infinite loop space machine takes as input topological categories.

We note that $ku_G$ and $ko_G$ are represented by the $G$-spaces which are the group completions of the monoids of equivariant bundles corresponding to split extensions

$$1 \to U_n \to U_n \times G \to G \to 1$$

(4.11)

and

$$1 \to O_n \to O_n \times G \to G \to 1,$$

(4.12)
respectively.

Consider the topological rings $\mathbb{C}$ and $\mathbb{R}$ with trivial $G$-action for any finite group $G$. Then, by definition, $K_G(\mathbb{C})$ and $K_G(\mathbb{R})$ are genuine $\Omega$-$G$-spectra with zeroth spaces given by the group completions of $\coprod BCat(\bar{G}, GL_n(\mathbb{C}))$, and $\coprod BCat(\bar{G}, GL_n(\mathbb{R}))$, respectively. By Theorem 3.3.8 and (4.10), these are the monoids of classifying spaces of $(G, U_n)$-bundles, and $(G, O_n)$-bundles, respectively, under Whitney sum. Note that here it was crucial that in the hypotheses of Theorem 3.3.8, even though the group of equivariance $G$ has to be discrete or finite, the structure group of the bundle is allowed to be compact Lie.

Therefore, we tautologically obtain the following theorem.

**Theorem 4.7.1.** Consider the topological rings $\mathbb{C}$ and $\mathbb{R}$ with trivial $G$-action for any finite group $G$. We have equivalences of connective $\Omega$-$G$-spectra

$$K_G(\mathbb{C}) \simeq ku_G \text{ and } K_G(\mathbb{R}) \simeq ko_G.$$ 

**4.7.2 Atiyah’s Real K-theory**

In [Ati66], Atiyah introduces a version of topological $K$-theory, called Real $K$-theory (with capital “R”) and denoted by $KR$, which he describes as a mixture of real $K$-theory $KO$ and equivariant topological $K$-theory $KU_G$ and $KO_G$. We have already shown that the connective versions $ko_G$ and $ku_G$ are instances of equivariant algebraic $K$-theory, and we show that so is $kr$. Therefore they all fit under the unifying framework of equivariant $K$-theory developed in this manuscript.

In the definition of $KR$, the bundles corresponding to split exact sequences (4.11) are replaced by equivariant bundles corresponding to split exact sequences

$$1 \to U_n \to U_n \rtimes C_2 \to C_2 \to 1,$$  

(4.13)
where the cyclic group of order 2, $C_2$, acts on $U_n$ by complex conjugation.

Atiyah shows a “Real” version of Bott periodicity, which gives that the representing space for $KR$ has deloopings with respect to $C_2$-representations, and thus $KR$ represents a genuine $\Omega$-$C_2$-spectrum. Of course, [Ati66] does not mention spectra and instead states the result in terms of a periodic $RO(C_2)$-graded cohomology theory.

By comparing the zeroth space of the connective spectrum $kr$ to the zeroth space of the equivariant algebraic $K$-theory spectrum for the topological ring $\mathbb{C}$ with $C_2$ conjugation action, we observe that they are $C_2$-equivalent, which yields the following theorem.

**Theorem 4.7.2.** Let $\mathbb{C}$ be the topological ring of complex numbers with conjugation action by $C_2$. The $RO(C_2)$-graded cohomology represented by the genuine $C_2$-spectrum $K_{C_2}(\mathbb{C})$ is isomorphic to Atiyah’s cohomology theory $kr$. 

CHAPTER 5
EQUIVARIANT K-THEORY OF EXACT AND WALDHAUSEN CATEGORIES

5.1 Introduction

In [DK82], Dress and Kuku define equivariant algebraic $K$-groups for exact categories with trivial $G$-action, which are later generalized by Kuku in [Kuk06] to Waldhausen categories with trivial $G$-action. The cases of interest to us will be categories which have a nontrivial $G$-action, and we want the definition of equivariant $K$-theory to see the action. We give a definition of the equivariant algebraic $K$-theory space of an exact or a Waldhausen $G$-category, which generalizes Kuku’s definition. We conjecture that the $K$-theory space we define is an infinite loop $G$-space, and we hope to be able to produce its deloopings in future work. Presumably, a “genuine” version of the additivity theorem will be needed for this.

5.2 Equivariant Q-Construction

An exact $G$-category $\mathcal{C}$ is a exact category with a $G$-action that respects the exact structure. Precisely, it is a functor $G \to \mathcal{E}xact\text{Cat}$, from $G$ to the category of exact categories and exact functors. Actually, by the following lemma, any action on an exact category $\mathcal{C}$ is necessarily by endomorphisms of exact categories.

Lemma 5.2.1. Suppose $\mathcal{C}$ is an exact category with $G$-action. Then the functor $g \cdot : \mathcal{C} \to \mathcal{C}$ is exact.

Proof. Note that we have natural isomorphisms

$$\text{Hom}_{\mathcal{C}}(g \cdot A, B) \cong \text{Hom}_{\mathcal{C}}(A, g^{-1} \cdot B)$$
and

$$\text{Hom}_\mathcal{C}(A, g \cdot B) \cong \text{Hom}_\mathcal{C}(g^{-1} \cdot A, B),$$

for any $g \in G$ and $A, B \in \text{Ob}(\mathcal{C})$. Thus $g \cdot$ is both a left and right adjoint.

The left exactness of the Hom functor, the Yoneda lemma and the adjunction isomorphisms give that a left adjoint functor is right exact and a right adjoint functor is left exact. We conclude that $g \cdot$ is an exact functor.

We analyze the fixed point category $\mathcal{C}^H$ for a subgroup $H$ of $G$.

**Theorem 5.2.2.** The fixed point category $\mathcal{C}^H$ for $H \subseteq G$ is an exact category with the exact sequences being the $H$-fixed exact sequences in $\mathcal{C}$.

This theorem is wrong as stated. The correct version should be:

**Theorem 5.2.2.** The homotopy fixed point category $\mathcal{C}^{hH}$ for $H \subseteq G$ is an exact category.

I thank Emanuele Dotto and Daniel Schäppi for illuminating discussions which led to the discovery of this mistake and its fix.

Before we prove the theorem we give the axiomatic definition of an exact category, which we will use. There is another definition of an exact category as a full subcategory of an abelian category which is closed under extensions, but that definition can be shown to be equivalent to this one [Qui73].

**Definition 5.2.3.** An exact category is an additive category $\mathcal{C}$ with a class $\mathcal{E}$ of short exact sequences

$$0 \to M' \to M \to M'' \to 0$$

satisfying the following axioms:

1. Any sequence in $\mathcal{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$.
2. \( \mathcal{E} \) contains the split exact sequences

\[
0 \to M' \to M' \oplus M'' \to M'' \to 0.
\]

3. The class of admissible monomorphisms (morphisms which appear as the first arrow in an exact sequence) is closed under composition and under base-change by arbitrary maps in \( \mathcal{E} \). Dually, the class of admissible epimorphisms (morphisms which appear as the second arrow in an exact sequence) is closed under composition and under cobase-change by arbitrary maps in \( \mathcal{E} \).

4. Admissible monomorphisms are kernels of their corresponding admissible epimorphisms, and admissible epimorphisms are the cokernels of their respective admissible monomorphisms.

Proof. (of Theorem 5.2.2)

I have removed the wrong proof. The point is that pushouts get preserved under the \( G \)-action only up to isomorphism, and the fixed point category is not closed under pushouts. However, after applying \( \text{Cat}(\widetilde{G}, -) \), it is possible to define the pushouts compatibly so that the fixed point categories are closed under pushouts.

We let \( Q\mathcal{C} \) stand for the Quillen \( Q \)-construction on \( \mathcal{C} \), which was introduced in [Qui73]. Now \( Q\mathcal{C} \) inherits a \( G \)-action from \( \mathcal{C} \). The following lemma is very easy, but the observation that the functor \( Q \) commutes with fixed points is crucial for the definitions to follow.

Lemma 5.2.4. For any subgroup \( H \subseteq G \), the fixed point categories \( (Q\mathcal{C})^H \) and \( Q(\mathcal{C}^H) \) coincide.

This is now true if \( \mathcal{C} \) is of the form \( \text{Cat}(\widetilde{G}, \mathcal{C}) \).

Note that if \( \mathcal{C} \) is an exact category, then \( \text{Cat}(\widetilde{G}, \mathcal{C}) \) is also an exact category with exact sequences of functors defined pointwise. More precisely, for \( F_1, F_2, F_3 \in \text{Cat}(\widetilde{G}, \mathcal{C}) \) and
\( \eta_1, \eta_2 \) natural transformations, the sequence

\[
F_1 \xrightarrow{\eta_1} F_2 \xrightarrow{\eta_2} F_3
\]

is an exact sequence in \( \text{Cat}(\tilde{G}, \mathcal{C}) \) if

\[
F_1(x) \rightarrow F_2(x) \rightarrow F_3(x)
\]

is an exact sequence in \( \mathcal{C} \) for all \( x \in \mathcal{C} \). This observation is also made in [DK82] and is what the definition of Dress and Kuku relies on. We give the following definition of the equivariant \( K \)-theory space for an exact \( G \)-category \( \mathcal{C} \).

**Definition 5.2.5.** The equivariant \( K \)-theory space of an exact \( G \)-category \( \mathcal{C} \) is

\[
K_G(\mathcal{C}) = \Omega BQ\text{Cat}(\tilde{G}, \mathcal{C})
\]

The equivariant \( K \)-groups for \( i \geq 0 \) are

\[
K_i^H(\mathcal{C}) = \pi_i^H(K_G(\mathcal{C})) \cong \pi_i(K_G(\mathcal{C})^H).
\]

Note that since the functors \( \Omega, B \) and \( Q \) commute with fixed points, the space \( K_G(\mathcal{C})^H \) is homotopy equivalent to \( \Omega BQ\text{Cat}(\tilde{G}, \mathcal{C})^H \).

We go on to show that if the \( G \)-action on \( \mathcal{C} \) is trivial, then our definition agrees with Dress and Kuku’s one. For a group \( G \) and a \( G \)-set \( X \), Dress and Kuku define the equivariant \( K \)-groups of an exact category \( \mathcal{C} \) with trivial \( G \)-action as

\[
K_i^G(X, \mathcal{C}) = K_i(\text{Cat}(\underline{X}, \mathcal{C})),
\]

where \( \underline{X} \) is the translation category of \( X \), and \( \text{Cat}(\underline{X}, \mathcal{C}) \) is the category of all functors with
G acting by conjugation. We reconcile this definition with the more general Definition 5.2.5.

**Lemma 5.2.6.** Let \( \mathcal{C} \) be an exact category with trivial \( G \)-action, and let \( H \) be a subgroup of \( G \). Then

\[
K^H_i(\mathcal{C}) \cong K^G_i(G/H, \mathcal{C}).
\]

**Proof.** We claim that we have an equivalence

\[
K_G(\mathcal{C})^H \simeq \Omega BQ \mathcal{Cat}(G/H, \mathcal{C}).
\]

Note that the left hand side is \( \Omega BQ \mathcal{Cat}(\tilde{G}, \mathcal{C})^H \), so it is enough to show we have an equivalence of exact categories

\[
\mathcal{Cat}(\tilde{G}, \mathcal{C})^H \simeq \mathcal{Cat}(G/H, \mathcal{C}).
\]

Since \( H/H \) is a full subcategory of \( G/H \), and any object in \( G/H \) is isomorphic to \( H \), the categories \( H/H \) and \( G/H \) are equivalent, and thus we have an equivalence of categories

\[
\mathcal{Cat}(G/H, \mathcal{C}) \simeq \mathcal{Cat}(H/H, \mathcal{C}).
\]

But \( H/H \) can be identified with \( H \), so we have

\[
\mathcal{Cat}(G/H, \mathcal{C}) \simeq \mathcal{Cat}(H, \mathcal{C}).
\]

Also, we have shown that \( \mathcal{Cat}(\tilde{G}, \mathcal{C})^H \simeq \mathcal{Cat}(H, \mathcal{C}) \) since the action on \( \mathcal{C} \) is trivial, so the claim follows.

The conclusion of the lemma follows immediately since

\[
K^H_i(\mathcal{C}) = \pi^H_i(K_G(\mathcal{C}))
\]

\[
\cong \pi_i(K_G(\mathcal{C})^H)
\]

\[
\cong \pi_i(\Omega BQ \mathcal{Cat}(G/H, \mathcal{C}))
\]

\[
\cong K_i(\mathcal{Cat}(G/H, \mathcal{C}))
\]

\[
= K^G_i(G/H, \mathcal{C}).
\]
5.3 Equivariant Plus=Q

Recall that in Definition 4.2.2, the equivariant algebraic $K$-theory space of a $G$-ring $R$ was introduced as a categorical model for the the equivariant group completion of the classifying space of the symmetric monoidal $G$-category $\mathcal{P}(R)$ of finitely generated projective modules and isomorphisms. The $G$-category $\mathcal{P}(R)$ is also exact, and we can take its equivariant $K$-theory space by using definition Definition 5.2.5. We now show that Definition 4.2.2 of the equivariant $K$-theory space for a $G$-ring $R$ agrees with Definition 5.2.5 of the equivariant $K$-theory space for the exact $G$-category $\mathcal{P}(R)$.

Note that even though we call the next theorem equivariant plus=Q, this is only symbolic because we do not have an equivariant version of the plus construction. However, the next theorem, combined with the harder Theorem 4.3.3, where we showed that the higher homotopy groups of $K_G(R)$ from Definition 4.2.2 are isomorphic to the higher homotopy groups of the group completion of the topological $G$-monoid of $(G, GL_n(R)_G)$-bundles, can be viewed as an equivariant version of the nonequivariant plus = Q theorem.

**Theorem 5.3.1** (Equivariant plus=Q). For a $G$-ring $R$, there is a weak $G$-equivalence

$$K_G(R) \simeq \Omega BQ\text{Cat}(\tilde{G}, \mathcal{P}(R)).$$

**Proof.** By definition, $K_G(R) = B(S^{-1}S)$ for $S = \text{Cat}(\tilde{G}, \text{iso } \mathcal{P}(R))$. We show that we have equivalences on fixed points

$$(BS^{-1}S)^H \simeq (\Omega BQ\text{Cat}(\tilde{G}, \mathcal{P}(R)))^H$$

for all $H \subseteq G$.

Note that since the functors $B$, $\Omega$, $S^{-1}$ and $Q$ commute with fixed points, this is equiv-
alent to requiring

\[ B(S^H)^{-1}S^H \simeq \Omega BQ\text{Cat}(\tilde{G}, \mathcal{P}(R))^H. \]

But this is true by the nonequivariant plus=Q theorem from [Qui73], since the category \( \text{Cat}(\tilde{G}, \mathcal{P}(R)) \) is split exact, and \( \text{isoCat}(\tilde{G}, \mathcal{P}(R)) = \text{Cat}(\tilde{G}, \text{iso}\mathcal{P}(R)) \).

\[ \square \]

### 5.4 Equivariant algebraic K-theory of Waldhausen categories

A Waldhausen \( G \)-category is a Waldhausen category \( \mathcal{C} \), with \( G \)-action that respects the Waldhausen structure. Precisely, it is a functor \( G \to \mathcal{WaldCat} \) from \( G \) to the category of Waldhausen categories and exact functors. This says that for any \( g \in G \) the functor \( g \cdot : \mathcal{C} \to \mathcal{C} \) preserves the zero object, cofibrations, weak equivalences and pushouts along cofibrations. Actually, since the functor \( g \cdot \) is both left and right adjoint to the functor \( g^{-1} \cdot \), it automatically preserves pushouts.

\[ \text{Lemma 5.4.1.} \quad \text{Let} \ \mathcal{C} \ \text{be a} \ G\text{-equivariant Waldhausen category, and let} \ H \ \text{be a subgroup of} \ G. \ \text{Then} \ \mathcal{C}^H \ \text{is a Waldhausen category with cofibrations and weak equivalences defined as the} \ H\text{-fixed cofibrations and weak equivalences in} \ \mathcal{C}. \]

Just as theorem 5.2.2, this theorem is wrong as stated. The fixed point categories are not closed under pushouts, but applying \( \text{Cat}(\tilde{G}, -) \) rectifies this. The correct version should be

\[ \text{Lemma 5.4.1.} \quad \text{Let} \ \mathcal{C} \ \text{be a} \ G\text{-equivariant Waldhausen category, and let} \ H \ \text{be a subgroup of} \ G. \ \text{Then the homotopy fixed point category} \ \mathcal{C}^{hH} \ \text{is a Waldhausen category.} \]

\[ \text{Lemma 5.4.2.} \quad \text{Let} \ \mathcal{C} \ \text{be a} \ G\text{-equivariant Waldhausen category and let} \ H \ \text{be a subgroup of} \ G. \ \text{Then} \ (S_\bullet \mathcal{C})^H = S_\bullet (\mathcal{C}^H). \]

Again, this is true if \( \mathcal{C} \) is of the form \( \text{Cat}(\tilde{G}, \mathcal{C}) \)
Note that if $\mathcal{C}$ is a Waldhausen category, then $\mathcal{C}at(\tilde{G}, \mathcal{C})$ is also a Waldhausen category with cofibrations and weak equivalences defined pointwise. More precisely, for $F_1, F_2 \in \mathcal{C}at(\tilde{G}, \mathcal{C})$,

$$F_1 \xrightarrow{\eta} F_2$$

is a cofibration or a weak equivalence if for every $x \in X$

$$F_1(x) \rightarrow F_2(x)$$

is a cofibration or a weak equivalence, respectively, in $\mathcal{C}$.

We define the equivariant algebraic $K$-theory space for a Waldhausen $G$-category $\mathcal{C}$ in the same spirit as the definitions for exact and symmetric monoidal categories, namely by replacing the category with $\mathcal{C}$ with $\text{Cat}(\tilde{G}, \mathcal{C})$ and applying the nonequivariant construction.

**Definition 5.4.3.** The equivariant algebraic $K$-theory space of a Waldhausen $G$-category $\mathcal{C}$ is

$$K_G(\mathcal{C}) = \Omega |S \cdot \text{Cat}(\tilde{G}, \text{iso } \mathcal{C})|.$$

Generalizing the previous definitions, the equivariant algebraic $K$-groups of a Waldhausen $G$-category are

$$K^H_i(\mathcal{C}) = \pi_i^H(K_G(\mathcal{C})).$$

Again, $K^H_i(\mathcal{C})$ is given by the homotopy groups of the fixed point space

$$K(\mathcal{C}, G)^H \simeq \Omega BQ\text{Cat}(\tilde{G}, \mathcal{C})^H.$$

The fixed point category $\text{Cat}(\tilde{G}, \mathcal{C})^H$ was described in detail in section 5.2.

It is not hard to see that the above definition agrees with Kuku’s definition of equivariant
algebraic $K$-groups of a Waldhausen category $\mathcal{C}$ with trivial $G$-action given in [Kuk06]:

$$K^G_i(X, \mathcal{C}) = K_i(\mathcal{C} at(X, \mathcal{C})),$$

where $X$ is the translation category of $X$. We claim that

$$K^H_i(\mathcal{C}) \cong K^G_i(G/H, \mathcal{C}),$$

and the proof of this statement goes through identically to the one of Lemma 5.2.6.

Now let $\mathcal{C}$ be an exact $G$-category. Then $\mathcal{C}$ is a Waldhausen $G$-category with cofibrations the admissible monomorphisms and weak equivalences the isomorphisms. Nonequivariantly, Waldhausen showed in [Wal85] that

$$BQ\mathcal{C} \simeq |S_{\bullet}\mbox{ iso }\mathcal{C}|.$$  \hspace{1cm} (5.1)

**Proposition 5.4.4** (Equivariant $Q = S_{\bullet}$). When $\mathcal{C}$ is an exact $G$-category, the equivalence (5.1) is a $G$-equivalence.

**Proof.** We have the nonequivariant equivalence (5.1) if we replace $\mathcal{C}$ with the nonequivariant exact category $\mathcal{C}^H$. Since all the constructions involved commute with fixed points, we get that

$$(BQ\mathcal{C})^H \simeq |S_{\bullet}\mbox{ iso }\mathcal{C}|^H$$

for all subgroups $H \subseteq G$, so that (5.1) is a weak $G$-equivalence. \hfill \Box

### 5.4.1 Beginning of equivariant $A$-theory

Let $X$ be a $G$-space. Let $R(X)$ be the Waldhausen category of retractive spaces over $X$, i.e., spaces $Y$ (not $G$-spaces!) with maps $X \overset{i}{\rightarrow} Y \overset{r}{\rightarrow} X$ such that $r \circ i = \text{id}$. Morphisms are continuous maps $Y \rightarrow Y'$ making the obvious diagram commute. Note that this category
inherits a $G$-action given by

$$g(X \xrightarrow{i} Y \xrightarrow{r} X) = X \xrightarrow{g^{-1}} X \xrightarrow{i} Y \xrightarrow{r} X \xrightarrow{g} X.$$  

For a morphism $f$, namely the data of a commuting diagram

![Diagram](image)

we define $(gf)(y) = f(y)$ for all $y \in Y$. This is indeed a morphism

$$g(X \xrightarrow{i} Y \xrightarrow{r} X) \rightarrow g(X \xrightarrow{i'} Y' \xrightarrow{r'} X),$$

since the diagram

![Diagram](image)

then also commutes.

The homotopy fixed points $R(X)^{hG} = \text{Cat}((\tilde{G}, R(X))^G$ are, by Theorem 2.3.9, spaces $Y$ together with isomorphisms $f(g) : Y \xrightarrow{\cong} Y$ for all $g$ making the following diagram commute:

![Diagram](image)
We also have that $f(e) = \text{id}$ and the cocycle condition $f(gh) = f(g)gf(h)$ is satisfied. We can define an action of $Y$ by $g \cdot y = f(g)(y)$. This is indeed an action since $f(e) = \text{id}$ and

\[
(gh) \cdot y = f(gh)(y) \\
= f(g)(gf(h)(y)) \\
= f(g)(f(h)(y)) \\
= g \cdot (h \cdot y).
\]

This action on $Y$ then makes $i$ and $r$ equivariant maps by the commutativity of the above diagram. Thus the category of homotopy fixed points $R(X)^{hG}$ can be identified with the category with objects $X \xrightarrow{i} Y \xrightarrow{r} X$, where $Y$ is a $G$-space and $i$ and $r$ are $G$-maps.

We can now define equivariant $A_G$-theory as the equivariant algebraic $K$-theory of the Waldhausen $G$-category $R(X)$, just as Waldhausen defines $A$-theory as the nonequivariant algebraic $K$-theory of the Waldhausen category $R(X)$.

### 5.4.2 Hopes and dreams

We make the following conjecture.

**Conjecture 5.4.5.** Let $\mathcal{C}$ be a Waldhausen $G$-category. Then the equivariant $K$-theory space $K_G(\mathcal{C})$ is an infinite loop $G$-space.

The real challenge is obtaining the conjectured genuine $\Omega G$-spectrum $\mathbb{K}_G(\mathcal{C})$, which deloops the space $K_G(\mathcal{C})$ with respect to all finite dimensional $G$-representations. Equivariant infinite loop space machines with the group completion property do not split exact sequences, and thus we need an equivariant version of the $S_\bullet$-construction.\(^1\)

---

\(^1\) A different approach is given by C. Barwick, who defines a $G$-spectrum from a $G$-Waldhausen category in the $\infty$-category setting in [Bar], using a model of $G$-spectra as diagrams of nonequivariant spectra inspired by [GMb].
Since an exact $G$-category is in particular a Waldhausen $G$-category, this will apply in particular to exact $G$-categories. Nonequivariantly, Waldhausen has constructed an $\Omega$-spectrum, which is easily seen to be a symmetric spectrum, by explicitly constructing deloopings of $\Omega|wS\mathcal{C}|$ by iterating the $S_\bullet$-construction.

T. Goodwillie suggested that for a $G$-manifold $M$, the equivariant $K$-theory spectrum $A_G(M)$ associated to the Waldhausen $G$-category $R(M)$, should split into the $G$-suspension spectrum of $M$ and a factor $Wh_G(M)$, which should be related to equivariant pseudoisotopies and $h$-cobordisms:

$$A_G(M) \simeq \Sigma^\infty_G M_+ \times Wh_G(M).$$

What a wonderful idea to end with!
CHAPTER 6
APPENDIX

6.1 Equivariant skeleta

Nonequivariantly, it is always assumed in $K$-theory that when we take the classifying space of a category which is not small, such as $\mathcal{P}(R)$, $\mathcal{F}(R)$, or $\text{Mod}(R)$, we are tacitly replacing the category by a small category, which is equivalent to it, such as its skeleton.

As we have seen in Section 2.7.4, the situation is a little trickier equivariantly, because we do not have an equivariant equivalence between a $G$-category and its skeleton. This is too much to hope for; however, we show that the discussion in Section 2.7.4, where we show that there is a weak $G$-equivalence $\text{Cat}(\widetilde{G}, \text{iso}\mathcal{F}(R)) \simeq \text{Cat}(\widetilde{G}, \mathcal{G}(R))$, generalizes. We show that for a $G$-category $\mathcal{C}$, we can put a $G$-action on the skeletal category $\text{sk}\mathcal{C}$, such that the inverse of the inclusion of the skeleton $i: \text{sk}\mathcal{C} \to \mathcal{C}$ is a $G$-map, which is a nonequivariant equivalence. This implies by Proposition 2.3.12 that the map

$$\text{Cat}(\widetilde{G}, \mathcal{C}) \longrightarrow \text{Cat}(\widetilde{G}, \text{sk}\mathcal{C})$$

is a weak $G$-equivalence. This suffices for our application, because in equivariant algebraic $K$-theory we are only taking classifying spaces of categories of the form $\text{Cat}(\widetilde{G}, \mathcal{C})$.

For an object $C \in \mathcal{C}$, denote by $C^{\text{rep}}$ the representative of the isomorphism class of $C$ in $\text{sk}\mathcal{C}$, so that if $C \cong D$, then $C^{\text{rep}} = D^{\text{rep}}$. We fix isomorphisms $\gamma_C: C \xrightarrow{\cong} C^{\text{rep}}$. The map

$$i^{-1}: \mathcal{C} \longrightarrow \text{sk}\mathcal{C}$$

is defined on objects as $C \mapsto C^{\text{rep}}$ and on morphisms as

$$(C \xrightarrow{f} D) \mapsto (C^{\text{rep}} \xrightarrow{\gamma_C^{-1}} C \xrightarrow{f} D \xrightarrow{\gamma_D} D^{\text{rep}}).$$
We define a $G$-action on $\text{sk}\ C$ in the following way. On objects,

\[ gC^{\text{rep}} := (gC)^{\text{rep}}. \]

We remark that there is no way to consistently pick the representatives such that $gC^{\text{rep}} = (gC)^{\text{rep}}$ is an equality in $\mathcal{C}$. However, we do have isomorphisms in $\mathcal{C}$

\[ (gC)^{\text{rep}} \xrightarrow{\cong} gC \xrightarrow{\cong} gC^{\text{rep}}. \]

We define the action on morphisms of $\text{sk}\ C$. We defined $g(C^{\text{rep}} \xrightarrow{f} D^{\text{rep}})$ as

\[ (gC)^{\text{rep}} \xrightarrow{\cong} gC \xrightarrow{\cong} gC^{\text{rep}} \xrightarrow{g\gamma D} gD \xrightarrow{\cong} (gD)^{\text{rep}}. \]

Now the map $i^{-1}$ is clearly equivariant on objects. We show it is also equivariant on morphisms. Let $f: C \xrightarrow{D} D^{\text{rep}}$ be a morphism in $\mathcal{C}$, which gets mapped by $i^{-1}$ to $C^{\text{rep}} \xrightarrow{\gamma_C^{-1}} C \xrightarrow{f} D \xrightarrow{\gamma_D} D^{\text{rep}}$ in $\text{sk}\ C$. Acting by $g$, we get

\[ (gC)^{\text{rep}} \xrightarrow{\cong} gC \xrightarrow{\cong} gC^{\text{rep}} \xrightarrow{g\gamma_C^{-1}} gD \xrightarrow{\cong} gD \xrightarrow{\gamma_D} (gD)^{\text{rep}}. \]

By composing the inverse isomorphism, this is the same as

\[ (gC)^{\text{rep}} \xrightarrow{\cong} gC \xrightarrow{gf} gD \xrightarrow{\gamma_D} (gD)^{\text{rep}}, \]

which is just $i^{-1}$ applied to $gC \xrightarrow{gf} gD$.

Therefore the map $i^{-1}: \mathcal{C} \to \text{sk}\ C$ is a $G$-map for the action we defined on $\text{sk}\ C$. 

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