Exercise 8.1 #14. Solve $\frac{dx}{dt} = 1 - 3x$, where $x(-1) = -2$.

Solution. We separate variables and integrate to obtain

$$\int \frac{dx}{1 - 3x} = \int dt = t + C$$

To compute the lefthand integral, we use $u$ substitution, with $u = 1 - 3x$, $du = -3 \, dx$:

$$\int \frac{dx}{1 - 3x} = -\frac{1}{3} \int \frac{du}{u} = -\frac{1}{3} \ln |u| = -\frac{1}{3} \ln |1 - 3x| = t + C$$

Now we multiply both sides by $-3$ and exponentiate both sides:

$$\ln |1 - 3x| = -3t + C_1$$

$$|1 - 3x| = e^{-3t+C_1} = e^{C_1}e^{-3t} = C_2e^{-3t}$$

(I use $C_1, C_2$, etc. whenever a new constant is introduced. The important thing to remember is that every $C_i$ is an arbitrary constant.) Now $|1 - 3x|$ is $\pm (1 - 3x)$, where the sign depends on the value of $x$. However, I can just incorporate this in the constant on the other side:

$$\pm (1 - 3x) = C_2e^{-3t}$$

$$1 - 3x = \pm C_2e^{-3t} = C_3e^{-3t}$$

$$x = \frac{1}{3}(1 - C_3e^{-3t}) = \frac{1}{3} + C_4e^{-3t}$$

Now we plug in the initial condition to find the value of the constant:

$$x(-1) = -2 = \frac{1}{3} + C_4e^{-3(-1)} = \frac{1}{3} + C_4e^{3}$$

$$-\frac{7}{3} = C_4e^{3}$$

$$C_4 = -\frac{7}{3}e^{3}$$

Thus the solution is

$$x(t) = \frac{1}{3} - \frac{7}{3}e^{-3t-3}$$
Exercise 8.1 #22. Denote by $L(t)$ the length of a fish at time $t$, and assume that the fish grows according to the von Bertalanffy equation

$$\frac{dL}{dt} = k(34 - L(t)) \text{ with } L(0) = 2$$

(a) Solve the differential equation.

(b) Use your solution in (a) to determine $k$ under the assumption that $L(4) = 10$. Sketch the graph of $L(t)$ for this value of $k$.

(c) Find the length of the fish when $t = 10$.

(d) Find the asymptotic length of the fish; that is, find $\lim_{t \to \infty} L(t)$.

Solution. (a) We follow the same steps as in problem 14:

$$\int \frac{dL}{34 - L} = \int kdt = kt + C$$

$$\int \frac{dL}{34 - L} = -\ln|34 - L| = kt + C$$

$$\ln|34 - L| = -kt + C_1$$

$$|34 - L| = e^{-kt+C_1} = C_2 e^{-kt}$$

$$34 - L = \pm C_2 e^{-kt} = C_3 e^{-kt}$$

$$L(t) = 34 - C_3 e^{-kt}$$

Plugging in the initial condition:

$$L(0) = 2 = 34 - C_3 e^{-k(0)} = 34 - C_3 e^0 = 34 - C_3$$

$$C_3 = 32$$

$$L(t) = 34 - 32e^{-kt}$$

(b) We just plug this condition in, as if it were another initial condition:

$$L(4) = 10 = 34 - 32e^{-k(4)} = 34 - 32e^{-4k}$$

$$32e^{-4k} = 24$$

$$e^{-4k} = \frac{24}{32} = \frac{3}{4}$$

$$-4k = \ln \frac{3}{4}$$

$$k = \frac{1}{4} \ln \frac{3}{4}$$

Here’s my sketch of $L(t)$ with this value of $k$: 2
(c) Simply plug in $t = 10$:

$$L(10) = 34 - 32e^{-1/4(ln 4/3)^{10}} = 34 - 32(e^{ln 4/3})^{-10/4} = 34 - 32\left(\frac{4}{3}\right)^{-5/2}$$

$$= 34 - 32\left(\frac{3}{4}\right)^{5/2} \approx 18.41$$

(d) We are computing the limit

$$\lim_{t \to \infty} 34 - 32e^{-1/4(ln 4/3)t}$$

To figure this out, we need to know whether $-\frac{1}{4} \ln \frac{4}{3}$ is positive or negative. Since $\frac{4}{3} > 1$, $\ln \frac{4}{3} > \ln 1 = 0$, so it turns out that $-\frac{1}{4} \ln \frac{4}{3}$ is negative. Therefore

$$\lim_{t \to \infty} 34 - 32e^{-1/4(ln 4/3)t} = 34 - 32(0) = 34$$

so the asymptotic length of the fish is 34.

Exercise 8.1 #48. Solve the equation $\frac{dy}{dx} = x^2 y^2$, with $y_0 = 1$ if $x_0 = 1$.

Solution. Separate variables and integrate as usual:

$$\int \frac{dy}{y^2} = \int x^2 dx$$

$$-\frac{1}{y} = \frac{1}{3} x^3 + C$$

Solving for $y$,

$$y = -\frac{1}{\frac{1}{3} x^3 + C} = \frac{-3}{x^3 + C_1}$$
Plugging in the initial condition,

\[
y_0 = 1 = \frac{-3}{1^3 + C_1} = \frac{-3}{1 + C_1}
\]

\[
1 + C_1 = -3
\]

\[
C_1 = -4
\]

Thus our solution is

\[
y = \frac{-3}{x^3 - 4}
\]

Exercise 8.1 #54. Consider the following differential equation, which is important in population genetics:

\[
a(x)g(x) - \frac{1}{2} \frac{d}{dx} [b(x)g(x)] = 0
\]

Here, \( b(x) > 0 \).

(a) Define \( y = b(x)g(x) \), and show that \( y \) satisfies

\[
\frac{a(x)}{b(x)} y - \frac{1}{2} \frac{dy}{dx} = 0
\]

(b) Separate variables in the above equation and show that if \( y > 0 \), then

\[
y = C \exp \left[ 2 \int \frac{a(x)}{b(x)} \, dx \right]
\]

Solution. (a) Setting \( y = b(x)g(x) \), we have that \( g(x) = \frac{y}{b(x)} \) since \( b(x) > 0 \). Thus, we can make this substitution to get

\[
a(x)g(x) - \frac{1}{2} \frac{d}{dx} [b(x)g(x)] = a(x) \frac{y}{b(x)} - \frac{1}{2} \frac{dy}{dx} \left[ b(x) \frac{y}{b(x)} \right] = a(x) \frac{y}{b(x)} - \frac{1}{2} \frac{dy}{dx} = 0
\]

which is what we wanted to show.

(b) Separating variables:

\[
\frac{a(x)}{b(x)} y = \frac{1}{2} \frac{dy}{dx}
\]

\[
2 \int \frac{a(x)}{b(x)} \, dx = \int \frac{dy}{y} = \ln |y| + C_1
\]

Since we are assuming that \( y > 0 \), this is

\[
\ln y + C_1 = 2 \int \frac{a(x)}{b(x)} \, dx
\]
\[
\ln y = -C_1 + 2 \int \frac{a(x)}{b(x)} \, dx
\]

\[
y = \exp \left[ -C_1 + 2 \int \frac{a(x)}{b(x)} \, dx \right] = C \exp \left[ 2 \int \frac{a(x)}{b(x)} \, dx \right]
\]

which is what we wanted to show.

**Aside:** It is not necessary to assume that \( y > 0 \) in this problem. If \( y < 0 \), then we would simply get a negative value of \( C \) instead of a positive one. If \( y = 0 \), then this equation still holds, but with \( C = 0 \). So in fact, no assumption on \( y \) is necessary at all. \( \square \)

**Exercise 9.1 #5.** Determine \( c \) such that

\[
\begin{align*}
2x - 3y &= 5 \\
4x - 6y &= c
\end{align*}
\]

has (a) infinitely many solutions and (b) no solutions. (c) Is it possible to choose a number for \( c \) so that the system has exactly one solution? Explain your answer.

**Solution.** Let’s begin by solving the system as much as we can. Subtracting 2 times the upper equation from the bottom equation we get

\[
\begin{align*}
2x - 3y &= 5 \\
0 &= c - 10
\end{align*}
\]

Thus, if \( c - 10 \) is not actually 0, then this system is inconsistent. That is to say, the system has no solutions if \( c \neq 10 \) this answers (b).

On the other hand, if \( c = 10 \), then our system is

\[
\begin{align*}
2x - 3y &= 5 \\
0 &= 0
\end{align*}
\]

It follows that this system has infinitely many solutions, so (a) holds exactly when \( c = 10 \).

Note that we have considered all possibilities: Either \( c = 10 \) or \( c \neq 10 \). In the former case, there are infinitely many solutions, and in the latter case, there are no solutions. There is no other possibility, so in particular, there is no value of \( c \) such that the system has exactly one solution. This answers (c). \( \square \)