Homework 2 Sample Solutions

provided by Kenneth Co

Extra Exercise 1. (a) Show that the following decomposition cannot hold

\[
\frac{x^3 + 10x^2 + 3x + 36}{(x - 1)(x^2 + 4)^2} = \frac{A}{x - 1} + \frac{Bx + C}{(x^2 + 4)^2}
\]

for constants A, B, and C.

(b) Show that the following decomposition cannot hold

\[
\frac{5x - 7}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{x - 1} + \frac{C}{x - 1}
\]

for constants A, B, and C.

Solution.

(a) Notice that when we add the right-hand side using their least common denominator, their sum will not have the term \(x^3\) in the numerator.

\[
\frac{A}{x - 1} + \frac{Bx + C}{(x^2 + 4)^2} = \frac{A(x^2 + 4)^2}{(x - 1)(x^2 + 4)^2} + \frac{(Bx + C)(x - 1)}{(x - 1)(x^2 + 4)^2}
\]

\[
= \frac{Ax^4 + 8Ax^2 + 16A}{(x - 1)(x^2 + 4)^2} + \frac{Bx^2 + Cx - Bx - C}{(x - 1)(x^2 + 4)^2}
\]

So for the following decomposition

\[
\frac{x^3 + 10x^2 + 3x + 36}{(x - 1)(x^2 + 4)^2} = \frac{A}{x - 1} + \frac{Bx + C}{(x^2 + 4)^2}
\]

to hold, we must have the following equality

\[
x^3 + 10x^2 + 3x + 36 = Ax^4 + 8Ax^2 + 16A + Bx^2 + Cx - Bx - C.
\]

However, the left-hand side has \(x^3\) with a coefficient of one, whereas the right-hand side has \(x^3\) with a coefficient of zero regardless of the constants A, B, and C. Thus, the decomposition cannot hold.
(b) Suppose the given decomposition is true for some constants $A$, $B$, and $C$. We will now look for a contradiction.

\[
\frac{5x - 7}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{x - 1} + \frac{C}{x - 1}
\]

\[
= \frac{A(x - 1)^2 + B(x - 1)^2 + C(x - 1)^2}{(x - 1)^3}
\]

\[
= (A + B + C)\frac{x - 1}{(x - 1)^3}
\]

\[
= (A + B + C)x - 2(A + B + C)x + (A + B + C)
\]

This tells us that $A + B + C = 0$, $2(A + B + C) = 5$, and $A + B + C = -7$. These clearly can’t be all true, so we have our contradiction. Thus, the decomposition cannot hold.

\[\square\]

Exercise 7.2 #36. (a) Use integration by parts to show that

\[
\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.
\]

(b) Apply the reduction formula in (a) repeatedly to compute

\[
\int x^3 e^x \, dx.
\]

Solution.

(a) We set $u = x^n$ and $dv = e^x$. Then, $du = nx^{n-1} \, dx$ and $v = e^x$. The desired integral follows from integration by parts.

(b) Using (a) several times, we get the following equations.

\[
\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx
\]

\[
= x^3 e^x - 3(x^2 e^x - 2 \int x e^x \, dx)
\]

\[
= x^3 e^x - 3x^2 e^x + 6(x e^x - \int e^x \, dx)
\]

\[
= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C
\]

\[
= e^x(x^3 - 3x^2 + 6x - 6) + C
\]

\[\square\]
Exercise 7.2 #64. Simplify the integrand and then use an appropriate substitution to evaluate
\[
\int \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2} \, dx.
\]

Solution.
\[
\int \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2} \, dx = \int \frac{(\sin x - \cos x)(\sin x + \cos x)}{(\sin x - \cos x)^2} \, dx
= \int \frac{\sin x + \cos x}{\sin x - \cos x} \, dx
\]
We make the substitution \( u = \sin x - \cos x \), to get \( du = (\sin x + \cos x) \, dx \) and
\[
\int \frac{\sin x + \cos x}{\sin x - \cos x} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\sin x - \cos x| + C.
\]

Exercise 7.3 #16. Use partial-fraction decomposition to evaluate the integral
\[
\int \frac{1}{(x-1)(x+2)} \, dx.
\]

Solution.
\[
\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}
= \frac{A(x+2)}{(x-1)(x+2)} + \frac{B(x-1)}{(x-1)(x+2)}
= \frac{Ax + 2A + Bx - B}{(x-1)(x+2)}
= \frac{(A+B)x + (2A-B)}{(x-1)(x+2)}
\]
This tells us that \( A + B = 0 \) and \( 2A - B = 1 \). Solving for their values, we get \( A = \frac{1}{3} \) and \( B = -\frac{1}{3} \). Finally, we compute for the integral.
\[
\int \frac{1}{(x-1)(x+2)} \, dx = \int \frac{1}{3(x-1)} \, dx - \int \frac{1}{3(x+2)} \, dx
= \frac{1}{3} (\ln |x-1| - \ln |x+2|) + C
\]

Exercise 7.3 #18. Use partial-fraction decomposition to evaluate the integral
\[
\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)} \, dx.
\]
Solution.

\[
\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}
\]

\[
= \frac{A(x+1)^2}{(x+1)^2(x-3)} + \frac{B(x-3)(x+1)}{(x+1)^2(x-3)} + \frac{C(x-3)}{(x+1)^2(x-3)}
\]

\[
= \frac{Ax^2 + 2Ax + A + Bx^2 - 2Bx - 3B + Cx - 3C}{(x+1)^2(x-3)}
\]

\[
= \frac{(A+B)x^2 + (2A-2B+C)x + (A-3B-3C)}{(x+1)^2(x-3)}
\]

This tells us that

\[
A + B = 4, \quad 2A - 2B + C = -1, \quad \text{and} \quad A - 3B - 3C = -1.
\]

Solving for \(A, B, C,\) and \(D,\) we get \(A = B = 2\) and \(C = -1.\) Finally, we compute for the integral.

\[
\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)} \, dx = \int \frac{2}{x-3} \, dx + \int \frac{2}{x+1} \, dx - \int \frac{1}{(x+1)^2} \, dx
\]

\[
= 2 \ln |x - 3| + 2 \ln |x + 1| + \frac{1}{x + 1} + C
\]