Exercise 1. Compute the following coefficients. You do not need to expand the entire expression.
(a) What is the coefficient of $a^4b^3$ in $(a + b)^7$?
(b) What is the coefficient of $x^4y^6$ in $(2y^3 + 5x^2)^4$?

Solution The binomial expansion is in general,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$  

This is what we will use to find the sought after coefficients.

(a) The coefficient of $a^4b^3$ in the expansion of $(a + b)^7$ is $\binom{7}{3}$. It accounts for the number of ways we can pick 4 copies of $a$ out of 7 possible copies of $a$ when we consider the product $(a + b)(a + b)(a + b)(a + b)(a + b)(a + b)(a + b)$.

Note that equivalently we can count the number of ways of picking 3 out of 7 copies of $b$ to combine, but

$$\binom{7}{3} = \binom{7}{4},$$

that’s an equivalent way of getting the answer.

Now

$$\binom{7}{3} = \frac{7!}{3!4!} = 5 \cdot 7 = 35,$$

so the coefficient of $a^4b^3$ in the expansion of $(a + b)^7$ is 35.

(b) The coefficient of $(2y^3)^k(5x^2)^{4-k}$ in the expansion of $(2y^3 + 5x^2)^4$ is $\binom{4}{k}$, for any $0 \leq k \leq 4$.

We are looking specifically for the coefficient of $x^4y^6$, and that is the term in the expansion corresponding to $k = 2$. The term in the expansion looks like

$$\binom{4}{2}(2y^3)^2(5x^2)^2 = \left(\frac{4!}{2!2!} \cdot 4 \cdot 25\right) x^4y^6 = 600x^4y^6.$$

So the coefficient of $x^4y^6$ is 600.