0.* Let \((X, \mathcal{T})\) and \((X', \mathcal{T}')\) be two topological spaces.

(i) Prove that the collection

\[ \mathcal{B} = \{[0] \times U | U \in \mathcal{T}\} \cup \{[1] \times U' | U' \in \mathcal{T}'\} \]

defines a basis for a topology on the disjoint union \(X \amalg X' := \{0\} \times X \cup \{1\} \times X'\).

(ii) If \(X\) and \(X'\) are not empty, the space \((X \amalg X', \mathcal{B}_3)\) is not connected.

(iii) \((X \amalg X', \mathcal{B}_3)\) is Hausdorff if and only if \((X, \mathcal{T})\) and \((X', \mathcal{T}')\) are.

(iv) \((X \amalg X', \mathcal{B}_3)\) is compact if and only if \((X, \mathcal{T})\) and \((X', \mathcal{T}')\) are.

1. Consider the topologies \(\mathcal{T}_{\text{disc}}, \mathcal{T}_{\text{codisc}}, \mathcal{T}_{\text{fin}}, \mathcal{T}_{\text{count}}, \mathcal{T}_{\text{st}}, \mathcal{T}_{\text{K}}, \mathcal{T}_{\text{up}}, \mathcal{T}_{\text{uplim}}\) on \([0, 1]\). For each topology,

   (i) find the connected component of \(\frac{1}{2}\) in \([0, 1]\), except for \(\mathcal{T}_{\text{K}}\).

   (ii) say if \([0, 1]\) is compact. You can assume that \(([0, 1], \mathcal{T}_a)\) is compact, which will be proven in class.

   (iii) say if \(K := \{\frac{1}{n+1} | n \in \mathbb{N}\}\) is compact.

   (iv) say if \(K \cup \{0\}\) is compact.

Justify each answer. To make the exercise shorter, it is useful to keep in mind that these topologies are related.

2. Find an example for every area of the following diagram. Justify your answers.

Hint: The disjoint union of topological spaces from Exercise 0 is useful to produce examples of spaces that are not connected.