Last week’s discussion on irrational rotations of $S^1$ introduced a new type of orbit behavior, where an orbit can densely fill a space. To continue this discussion, we will need to name this type of behavior and try to characterize why some dynamical systems behave this way. We start today with a few definitions.

**Definition 1.** A set $Y \subset X$ is invariant under a map $f : X \to X$, if

$$f \middle|_Y : Y \to Y.$$

**Definition 2.** A homeomorphism $f : X \to X$ is called *topologically transitive* if $\exists x \in X$ such that $O_x$ is dense in $X$. An non-invertible map is called topologically transitive if $\exists x \in X$ such that $O^*_x$ is dense in $X$.

**Definition 3.** A homeomorphism $f : X \to X$ is *minimal* if $\forall x \in X O_x$ is dense in $X$ (the forward orbit is dense for a noninvertible map).

**Definition 4.** A closed, invariant set is *minimal* is there does not exist a proper, closed invariant subset.

More notes:

- Like in the case of open and closed domains in vector calculus, a set is closed if it contains all of its limit points. And for any set $X$, the closure of $X$, denoted $\overline{X}$ is defined to be the closed set obtained by adding to $X$ all of its limit points (think of adding the sphere which is the boundary of an open ball in $\mathbb{R}^3$). In the case of a minimal map $f : X \to X$, for any $x \in X$, we have $\overline{O_x} = X$.
- Same is true for a topologically transitive map $f$, if one takes any point on the dense orbit.
- Irrational rotations of the circle are minimal!!

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0.1. **Application: Periodic Function Reconstruction via Sampling.** Consider the two functions in the picture.

- Each is periodic and of the same period as the other.
- Each can be viewed as a real-valued smooth function on $S^1$. And each takes values in the interval $I = [-1, 1]$.
- Question: Are the values of these two functions equally distributed equally (or even evenly) on $I$?
- Question: If we knew the period and range of some unknown function, and needed to sample the function (create a sequence of function values) to see which of the above two function was the one we are seeking, how can we design our sampling to ensure we can differentiate between these two?

Dynamics attempts to answer this question. Let $\{x_n\}$ be a sequence (think of this sequence as a sampling of the function), and $a < b$ two real numbers. Define

$$F_{a,b}(n) = \# \left\{ k \in \mathbb{Z} \mid 1 \leq k \leq n, a < x_k \leq b \right\}$$

as the number of times the sequence up to element $n$ visits the interval $(a, b) \subset \mathbb{R}$. Really, this is the same definition of $F$ as before on the arc $\Delta \subset S^1$. The only change in this case is that we are defining $F$ in this context as an interval in $\mathbb{R}$. Then define the *relative frequency* in the same way as before. In the figure, the relative frequency of $\{x_n\}$ on the interval $(a, b]$ shown is

$$\frac{F_{a,b}(n)}{n} \bigg|_{n=6} = \frac{2}{6} = \frac{1}{3}.$$  

We say that $\{x_n\}$ has an *asymptotic distribution* if $\forall a, b$, where $-\infty \leq a < b \leq \infty$, the quantity

$$\lim_{n \to \infty} \frac{F_{a,b}(n)}{n}$$

exists. In a sense, we are defining the percentage of the time that a sequence visits a particular interval.
In the case where the sequence has an asymptotic distribution, the function
\[ \Phi_{\{x_n\}}(t) = \lim_{n \to \infty} \frac{F_{-\infty,t}(n)}{n} \]
is called the distribution function of the sequence \( \{x_n\} \). Here \( \Phi \) is monotonic, and measures how often the values of a sequence visit regions of the real line as one varies the height of an interval \((-\infty, t]\).

**Definition 5.** A real-valued function \( \varphi \) on a closed, bounded interval is called piecewise monotonic if the domain can be partitioned into finite many subintervals on which \( \varphi \) is monotonic. A real-valued function on \( \mathbb{R} \) is piecewise monotonic if it is piecewise monotonic on every closed, bounded subinterval of \( \mathbb{R} \).

**Remark 6.** Monotonic means strictly monotonic here. Really, this means that there are no flat (purely horizontal on an open interval) regions of the graph of \( \varphi \). Think of functions like \( f(x) = \sin x \), and polynomials of degree larger than 1, which are piecewise monotonic, and functions like
\[
g(x) = \begin{cases} 
-(x + 2)^2 & -4 \leq x < -2 \\
0 & -2 \leq x \leq 0 \\
x^2 & 0 < x \leq 2
\end{cases}
\]
which is not piecewise monotonic (See the graph of \( g(x) \) below).

When \( \varphi \) is piecewise monotonic, the pre-image of any interval \( I \) is a finite union of intervals in the domain (see the figure).

**Definition 7.** The \( \varphi \)-length of an interval \( I \) is
\[ \ell_{\varphi}(I) := \ell(\varphi^{-1}(I)) \]  
- This is the total length of all pieces of the domain that map onto \( I \). In the figure, \( \ell_{\varphi}(I) = \ell(A) + \ell(B) \).
- For piecewise monotonic functions \( \varphi \), the \( \varphi \)-length is a continuous function of the end points of \( I \) (vary one end point of \( I \) continuously, and the \( \varphi \)-length of \( I \) also varies continuously. This doesn’t work with flat regions since the \( \varphi \)-length \( \ell_{\varphi} \) would then jump as one hits the value of the flat region.

Indeed, let’s look at the \( g(x) \) in the figure more closely. Here, one can calculate the \( \varphi \)-length. Indeed, choose the interval \( I = [-4, t] \). Here, \( t \) is the function value, and there is only a single interval mapped onto \( i \) for any value of \( t \).

For \( t < 0 \), this interval is given in the figure as the interval of the domain \( g^{-1}(I) = [-4, r] \), where \( g(r) = t \). Solving the equation \( g(r) = t \) for \( r \) yields
\[ -(r + 2)^2 = t \iff r = -\sqrt{-t} - 2 \]
where we chose the negative branch of the square root function in the middle step to account for the domain restrictions. Here, the $g$-length of $I$,
\[
\ell_g(I) = \ell \left( g^{-1}([−4, t]) \right) = -2 - \sqrt{-t} - (-4) = 2 - \sqrt{-t}.
\]

Now for $t > 0$, the same calculation yields $\ell_g(I) = 4 + \sqrt{t}$ for $I = [-4, t]$. Putting these two pieces of the $g$-length function together yields the graph of
\[
\ell_g(I) = \begin{cases} 
2 - \sqrt{-t} & -4 \leq t < 0 \\
4 + \sqrt{t} & 0 < t \leq 4
\end{cases}
\]
which has a jump discontinuity at $t = 0$. In fact, the only way to change $g(x)$ to make the $g$-length function continuous is to remove the middle piece of the $g(x)$ function and translate one or the other pieces right or left to again make $g(x)$ continuous. But that would have the effect of moving the two pieces of the graph of $\ell_g(I)$ together. The jump discontinuity becomes a hole in the graph, easily filled. But in this case, the changed $g(x)$ has been made piecewise monotone!

One can show that for a piecewise monotonic function $\varphi$, a distribution function for $\varphi$ is
\[
\Psi : \mathbb{R} \to \mathbb{R}, \quad \Psi_\varphi(t) = \ell_\varphi((-\infty, t)).
\]
We can use this for:

**Theorem 8.** Let $\varphi$ be a $T$-periodic function of $\mathbb{R}$ such that $\varphi_T = \varphi|_{[0,T]}$ is piecewise monotone. If $\alpha \notin \mathbb{Q}$ and $t_0 \in \mathbb{R}$, then the sequence $x_n = \varphi(t_0 + n\alpha T)$ has an asymptotic distribution with distribution function
\[
\Phi_{(x_n)}(t) = \frac{1}{T} \Psi_\varphi(t) = \frac{\ell(\varphi_{-1}((-\infty, t)))}{T}.
\]

We won’t prove this or study it in any more detail. But there is an interesting conclusion to draw from this. In the theorem, the sequence of samples of the $T$-periodic function $\varphi$ has the same distribution function as the actual function $\varphi$, (defined over the period, that is) precisely when the sampling is taken at a rate which is an irrational multiple of the period $T$. In this way, the sequence, over the long term, will fill out the values of $\varphi$ over the period in a dense way. In a way, one can recover the function $\varphi$ from a sequence of regular samples of it only if the sampling is done in a way which ultimately allows for all regions of the period to be visited evenly. This is a very interesting result.
In the book is an actual calculation of the distribution function for the sequence \( \{ \sin n \} \). Since the natural numbers are not a rational multiple of \( 2\pi \), the period of the sine function, this distribution function is precisely the same as that distribution function of the smooth function \( f(x) = \sin x \), defined on the interval \([0, 2\pi]\). Take a good look at this example.