1. COUNTING PERIODIC ORBITS

It seems like we spend a lot of time in our study of dynamical systems on the classification and counting of periodic orbits of a map $f : X \to X$. To understand why, consider

- $n$-periodic points are the fixed points of the map $f^n$.
- Periodic points, like fixed points, have stability features.
- There are existence theorems for periodic points.
- Many times, we can “solve” for them, without actually solving the dynamical systems.

**Definition 1.** For $f : X \to X$ a map, let

$$\text{Per}_n(f) := \{ x \in X | f^n(x) = x \}$$

be the set of all $n$-periodic points of $f$ and $P_n(f) := \#\text{Per}_n(f)$. Also let

$$\text{Per}(f) := \bigcup_{n \in \mathbb{N}} \text{Per}_n(f).$$

Note that $\text{Per}_n(f)$ also includes all $m$-periodic point when $m|n$. In particular, the 1-periodic points are the fixed points and these are in $\text{Per}_n(f)$ for all $n \in \mathbb{N}$.

As a sequence, $\{ P_n(f) \}_{n \in \mathbb{N}}$ can say a lot about $f$.

Recall $E_2 : S^1 \to S^1$, $E_2(z) = z^2$, $z = e^{2\pi i x} \in S^1$, or $E_2(s) = (2s \mod 1)$, for $s \in S^1$, depending on your model for $S^1$.

**Proposition 2.** $P_n(E_2) = 2^n - 1$, and all periodic points are dense in $S^1$ (i.e., $\overline{\text{Per}(E_2)} = S^1$).

**Proof.** Using the model $E_2(z) = z^2$, we find that $z$ is an $n$-periodic point if

$$\left( \ldots \left( (z^2)^2 \right) \ldots \right)^2 = z \text{ or } z^{2n} = z \text{ or } z^{2^n - 1} = 1.$$ 

Thus every periodic point is an order-$(2^n - 1)$ root of unity (and vice versa). And there are exactly $2^n - 1$ of these, uniformly spaced around the circle. In fact, to any rational $\frac{p}{q} \in \mathbb{Q}$, the point $e^{2\pi i \left( \frac{p}{q} \right)}$ is a $q$th root of unity. If $q = 2^n - 1$, for $n \in \mathbb{N}$, then $e^{2\pi i \left( \frac{4}{7} \right)}$ is an order-$n$ fixed point. Now as $n$ goes to $\infty$, the spacing between order-$(2^n - 1)$ roots of unity goes to 0. Hence any point $x \in S^1$ can be written as the limit of a sequence of these points. Hence will be in the closure of $P(E_2)$. \qed

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**Figure 1.** Order-$n$ fixed points of $E_2$ spaced evenly in $S^1$

**Example 3.** $z$ is 2-periodic if $z^{2^2-1} = z^3 = 1$. These are the points $z = e^{2\pi i (\frac{k}{3})}$, for $k = 1, 2$. To see how this works,

$$E_2(e^{2\pi i (\frac{1}{3})}) = \left(e^{2\pi i (\frac{1}{3})}\right)^2 = e^{2\pi i (\frac{2}{3})}, \text{ while }$$

$$E_2(e^{2\pi i (\frac{2}{3})}) = \left(e^{2\pi i (\frac{2}{3})}\right)^2 = e^{2\pi i (\frac{4}{3})} = e^{2\pi i (\frac{1}{3})}.$$ 

We can calculate the growth rate of $P_n(E_2)$ in the obvious way: Define the *truncated* natural logarithm

$$\ln^+_x = \begin{cases} 
\ln x & x \geq 1 \\
0 & \text{otherwise}
\end{cases}.$$ 

Then define $p(f) = \lim_{n \to \infty} \frac{\ln^+_n P_n(f)}{n}$ as the relative logarithmic growth of the number of $n$-periodic points of $f$ with respect to $n$.

For our case, then, where $E_2(z) = z^2$,

$$p(E_2) = \lim_{n \to \infty} \frac{\ln^+_n (2^n - 1)}{n} = \lim_{n \to \infty} \frac{\ln^+_n 2^n (1 - 2^{-n})}{n}$$

$$= \lim_{n \to \infty} \frac{\ln^+_n 2^n + \ln^+_n (1 - 2^{-n})}{n} = \ln 2.$$ 

This is the exponential growth rate of the periodic points of the map $E_2$. Note that the growth factor is 2 at each stage, hence the exponential growth rate is the exponent of $e$ which corresponds to the growth factor. Here $2 = e^{\ln 2}$.

**Proposition 4.** For $f : S^1 \to S^1$, $f(z) = z^m$, where $m \in \mathbb{Z}$ and $|m| > 1$,

$$P_n(f) = |m^n - 1|,$$

the set of all periodic points is dense in $S^1$, and $p(f) = \ln |m|$.
Exercise 1. Show this for $m = -3$.

Here is an interesting fact: Let $f(z) = z^2$. The image of any small arc in $S^1$ is twice as long as the original arc. However, there are actually 2 disjoint pre-images of each small arc, and each is exactly half the size. Combined, the sum of the lengths of these two pre-images exactly matches the length of the image. Thus this expanding map on $S^1$ actually preserves length! Some notes about this:

- This is actually true for all of the expanding maps $E_m : S^1 \to S^1$, $E_m(z) = z^m$, where $m \in \mathbb{Z}$, and $|m| > 1$.
- This is a somewhat broadening of the idea of area preservation for a map. When the map is onto but not 1-1 (in this case, the map is 2-1), the relationship between pre-image and image is more intricate, and care is needed to understand the relationship well.

1.1. The quadratic family. For $\lambda \in \mathbb{R}$, let $f_\lambda : \mathbb{R} \to \mathbb{R}$, where $f_\lambda(x) = \lambda x(1 - x)$ is also called the logistic map on $\mathbb{R}$. For $\lambda \in [0, 4]$, we can restrict to $I = [0, 1]$, and $f_\lambda : I \to I$ is the interval map family we partially studied already. In fact, we can summarize our results so far: For $\lambda \in [0, 3]$, the dynamics are quite simple. There are only fixed points, and no nontrivial periodic points, and all other points are asymptotic to them. The fixed points are at $x = 0$ and $x = 1 - \frac{1}{\lambda}$.

Some new facts:

1. For $\lambda \in [3, 4]$, a LOT happens! (we will get to this later in the course.)
2. For $\lambda > 4$, $I$ is not invariant.
3. Since $f_\lambda$ is quadratic, $f^n_\lambda$ is at most of degree $2^n$. Thus the set of $n$-periodic points must be solutions to the equation $f^n_\lambda(x) = x$. Bringing $x$ to the other side of the equation, the set $P_n(f_\lambda)$ must consists of the roots of an (at most) $2^n$-degree polynomial. Hence $P_n(f_\lambda) \leq 2^n$, for all $\lambda \in \mathbb{R}$.
4. For $\lambda > 4$, many points escape the interval $I$. However, as we will see, many points have orbits which do not. We can still talk about the map on the set of all of these points....

Let $\lambda > 4$, and consider the first iterate of $f_\lambda$. Notice (see the figure), that the intervals $I_1$ and $I_2$ are both mapped onto $[0, 1]$ and that each contains exactly one fixed point. Under the second iterate of the map, $f^2_\lambda$, only points in the 4 intervals $J_i$, $i = 1, 2, 3, 4$ remain in $[0, 1]$. Here there are 4 fixed points (again one in each interval). But notice that only two of them are new, $y_1$ and $y_2$. These two new ones are period-2 points that are not fixed points. See in the cobwebbed figure the period-2 orbit on the right of the figure.

Continue iterating in this fashion, and one can see that there will be
The map $f_\lambda$, $f_\lambda^2$, and the period-2 orbit

- $2^n$ intervals of points that remain in $I$ after $n$-iterates.
- The next iterate of $f_\lambda$ maps each of these $2^n$ intervals onto $[0,1]$, creating a single fixed point in each interval (of $f_\lambda^n$).
- You can see then (and one can prove this by induction) that $P_n(f_\lambda) = 2^n$, when $\lambda > 4$.

**Exercise 2.** Show that for $f_\lambda$ where $\lambda > 4$, if $O_x^+ \notin I$, then $O_x \rightarrow -\infty$. Also show that once an iterate of $x$ under $f_\lambda$ leaves $I$, it never returns.

So what about the points whose orbits stay in $I$? We can construct this as follows: For $x \in I$, call $O_x^n$ the $n$th partial (positive) orbit of $x$, where

$$O_x^n = \{ y \in I \mid y = f^i(x), \ i = 0, \ldots, n-1 \}.$$

Then define

$$C_n = \{ x \in I \mid O_x^n \in I \}.$$

Then $C_0 = I$, $C_1 = I_1 \cup I_2$, $C_2 = J_1 \cup J_2 \cup J_3 \cup J_4$, and $C_n \subset C_{n-1}$ for all $n \in \mathbb{N}$. Finally, define

$$C = \bigcap_{n=0}^{\infty} C_n,$$

Then $f_\lambda : C \rightarrow C$ is a discrete dynamical system.

What does this set $C$ look like? For starters, it seems quite similar in construction to our Canter Ternary Set. Be careful here, though. The connected subintervals of $C_n$ will not always be the same length in $C_n$. You can see this in the above figure, but should also check specifically for $C_2$. It should be certain that if $x \in I$ is $n$-periodic, then $x \in C$. But are these the only points whose entire orbit lies in $I$? What about a point $y \in [0,1]$ which is well-approximated by periodic points? This means that there is a sequence of periodic points in $I$ which converges to $y$. Is that enough to ensure that $O_y \in [0,1]$? This is an important question (which should be yes, by continuity.) It turns out that there are a lot of non-periodic points in $C$. In fact, there are an uncountable number. In fact, a Cantor’s set-worth! To see this, we need a better definition of a Cantor Set than what comes from our Canter Ternary Set above.
Definition 5. A non-empty subset of $I$ is called a Cantor Set if it is a closed, totally-
disconnected, perfect subset of $I$.

Definition 6. A non-empty subset $C \subset I$ is perfect if, for every point $x \in C$, there exists a
sequence of points $x_i \in C$, $i \in \mathbb{N}$, where $\{x_i\}_{i \in \mathbb{N}} \to x$.

Definition 7. A non-empty subset $C \subset I$ is totally-disconnected if, for every $x, y \in C$, the
closed interval $[x, y] \notin C$.

Roughly, there are no isolated points in a perfect set. And there are no closed, positive-
length intervals in a totally disconnected subset of an interval.

Proposition 8. Let $f_\lambda : I \to \mathbb{R}$ be defined by $f_\lambda(x) = \lambda x (1-x)$, where $\lambda > 4$ and let

$$C = \left\{ x \in I \mid \mathcal{O}_x \subset I \right\}.$$  

Then $C$ is a Cantor Set and $f_\lambda |_C$ is a discrete dynamical system.

Proof. By the exercise above, we already know that all periodic points are in $C$. For the
moment, let’s consider only the case that $\lambda > 2 + \sqrt{5} > 4$. In this case, we are assured that
$|f_\lambda'(x)| > \mu > 1$, $\forall x \in C_1$ and some number $\mu$.

Exercise 3. Verify this fact.

And hence, by the Chain Rule, we have $|f_\lambda'(x)| > 1$, $\forall x \in C$. Hence we know $f_\lambda$ is
expanding. Since $C$ is an arbitrary intersection of closed sets, it is certainly closed.

As for totally-discontinuous, let’s assume that for $x, y \in C$, where $x \neq y$, the interval
$[x, y] \in C$. Then the orbit of the entire interval lies completely in $C$. But since $f_\lambda$ is
expanding, $|f(x) - f(y)| > \mu|x - y|$. And for each $n \in \mathbb{N}$, $|f^n(x) - f^n(y)| > \mu^n|x - y|$. Choose
$n > \frac{\ln|x - y|}{\ln\mu}$. Then $|f^n(x) - f^n(y)| > 1$. But then $f^{n+1}([x, y]) \notin C$. This contradiction means
that no positive-length intervals exist in $C$, and establishes that $C$ is totally discontinuous.

To see that $C$ is perfect, assume for a minute that there exists an isolated point $z \in C$. Being isolated means that there is a small open interval $U(z) \subset I$, where for all $x \in U(z)$,
where $x \neq z$, we have $x \notin C$. Now, since $z \in C$, it is in a subinterval of every $C_n$. For any
choice of $n \in \mathbb{N}$, call the interval $[x_n, y_n] \in C_n$ where $z \in [x_n, y_n]$. Create a sequence of nested
closed intervals $\{[x_i, y_i]\}_{i \in \mathbb{N}}$, where for every $i$, $z \in [x_i, y_i] \subset C_i$. Each endpoint $x_i$ is eventually
fixed and hence $x_i \in C$ for all $i \in \mathbb{N}$. But $C$ is totally disconnected. Hence the intersection

$$\bigcap_{i=1}^{\infty}[x_i, y_i]$$

can only consist of one point, and $z$ is in this set. Thus, as a sequence $\{x_i\}_{i \in \mathbb{N}} \to z$, and $z$
is NOT isolated in $C$. Hence $C$ is perfect, and hence $C$ is a Canter Set.
As a final note, we will relegate a discussion of why $C$ is still a Canter set when $4 < \lambda < 2 + \sqrt{5}$ to the following remark, noting that the proof requires a subtle bit of finesse not totally germane to the current discussion.

\[ \square \]

Remark 9. When $4 < \lambda < 2 + \sqrt{5}$, the map $f_\lambda$ is not expanding on $C_1$. Indeed, for $\epsilon > 0$, let $\lambda = 4 + \epsilon$. Then the first intersection of the graph of $f_\lambda$ and the $y = 1$ line is at $x_1 = \frac{1}{2 \lambda} \left( 1 - \sqrt{1 - \frac{4}{\lambda}} \right)$. The derivative of $f_\lambda$ at this crossing is $f_\lambda'(x_1) = \sqrt{\lambda^2 - 4\lambda}$, which evaluates (when $\lambda = 4 + \epsilon$, see the figure below, to

\[ f_\lambda'(x_1) = \sqrt{\lambda^2 - 4\lambda} = \sqrt{4\epsilon + \epsilon^2} > 2\sqrt{\epsilon}. \]

The derivative of the square of the map at $x_1$ has a much higher derivative since the derivative of the image of $x_1$ is $-\lambda = -(4 + \epsilon)$ at the image point $f_\lambda(x_1) = 1$. Hence the derivative of the square of this map is greater than $2\sqrt{\epsilon}$. This happens all though the interval, and the map can be said to be eventually expanding, in that $\exists N \in \mathbb{N}$ where for all $n > N$ the map $f_\lambda^n(x)|_{C_n}$ is expanding. Then the proof above holds. Thus the proposition is true for all $\lambda > 4$.

This quadratic family is an example of a unimodal map: A continuous map defined on an interval that is increasing to the left of an interior point and decreasing thereafter.

**Proposition 10.** Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1) = 0$ and suppose there exists $c \in (0, 1)$ such that $f(c) > 1$. Then $P_n(f) \geq 2^n$. If, in addition, $f$ is unimodal and expanding, then $P_n(f) = 2^n$.

**Definition 11.** A map $f : [0, 1] \to \mathbb{R}$ is expanding if

\[ \left| f(x) - f(y) \right| > |x - y| \]

on each interval of $f^{-1}([0, 1])$.

Examples of expanding maps include the logistic map for suitable values of $\lambda > 4$, and the circle maps $E_m$, where $m \in \mathbb{Z}$ and $|m| > 1$ (you should modify the definition here to include maps of $S^1$). Note here:

- In the Proposition, the condition $f(0) = f(1) = 0$ and continuity ensure that the map will “fold” the image over the domain, and
- the condition $f(c) > 1$ ensures the folding will be complicated, with lots of points escaping, while lots of points will not.
1.2. Expanding Maps. Here is a better definition of an expanding map (albeit limited now to circle maps):

**Definition 12.** A $C^1$-map $f : S^1 \to S^1$ is expanding if $|f'(x)| > 1$, $\forall x \in S^1$.

**Example 13.** It should be obvious by this definition that the map $E_m$, where $m \in \mathbb{Z}$ and $|m| > 1$ is expanding, since $E_m(x) = mx \mod 1$ is differentiable and $|E'_m(x)| = |m| > 1$ for all $x \in S^1$.

Recall that the degree of a circle map is a well defined property that measures how many times the image of a map of $S^1$ winds itself around $S^1$.

**Lemma 14.** Let $f, g : S^1 \to S^1$ be continuous. Then

$$\deg(g \circ f) = \deg(g) \deg(f).$$

**Proof.** Degree is defined via a choice of lift: Given lifts $F, G : \mathbb{R} \to \mathbb{R}$ of these two maps, we have for $s \in S^1$ and $k \in \mathbb{Z}$,

$$G(s + k) = G(s + k - 1) + \deg(g) = G(s + k - 2) + 2 \deg(g) = \cdots = G(s) + k \deg(g).$$

But this means

$$G(F(s + 1)) = G(F(s) + \deg(f)) = G(F(s)) + \deg(f) \deg(g).$$

□

**Example 15.** $\deg(f^n) = (\deg(f))^n$.

Hence we can use this to show:

**Proposition 16.** If $f : S^1 \to S^1$ is expanding, then $|\deg(f)| > 1$ and $P_n(f) = |(\deg(f))^n - 1|$.

Here is a 2-dimensional version of periodic point growth. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x, y) = (2x + y, x + y)$. We can also write $L$ as the linear vector map

$$L(x) = Ax,$$

where $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

We know that since $A$ has integer entries, it takes integer vectors to integer vectors, and hence descends to a map on the two torus $\mathbb{T}^2$. Indeed, if $x_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ satisfy $x_1 - x_2 \in \mathbb{Z}^2$, then

$$L(x_1 - x_2) = L(x_1) - L(x_2) \in \mathbb{Z}^2.$$

But then $L(x_1) - L(x_2) = 0 \mod 1$, which means $L(x_1) = L(x_2) \mod 1$. Hence the map $L$ induces a map on $\mathbb{T}^2$ which assigns

$$(x, y) \mapsto (2x + y \mod 1, x + y \mod 1).$$
We will call this new induced map on the torus $F_L : T^2 \to T^2$, where
\[
F_L(x) = Ax, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad x \in T^2.
\]

Some notes:

- This map is an automorphism of $T^2$: A homeomorphism that preserves also the ability of points on the torus to be added together (multiplied, if one defines the multiplication correctly).
- $F_L$ is also invertible since it is an integer matrix of determinant 1. The inverse map $F^{-1}_L : T^2 \to T^2$ is given by the matrix $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.
- The eigenvalues of $F_L$ (really the eigenvalues of $A$) are the solutions to the quadratic equation $\lambda^2 - 3\lambda + 1 = 0$, or
\[
\lambda = \frac{3 \pm \sqrt{5}}{2}.
\]

Note that
\[
\lambda_1 = \frac{3 + \sqrt{5}}{2} > 1, \quad \text{and} \quad \lambda_2 = \frac{3 - \sqrt{5}}{2} < 1,
\]
so that the matrix defining $F_L$ is a hyperbolic matrix (determinant-1 with eigenvalues off the unit circle in $\mathbb{C}$).

**Question 17. How does $F_L$ act on $T^2$?**

Really the answer to this question relies on how $L$ acts on $\mathbb{R}^2$. Watching the model of $T^2$ as the unit square in $\mathbb{R}^2$ as it is acted on by $L$ provides the means to study the $F_L$ action on $T^2$. This is the two dimensional version of studying a lift of a circle map on $\mathbb{R}$ as a means of studying the circle map.

Linear maps of the plane take lines to lines. Hence they take polygons to polygons, and, in this case, they take parallelograms to parallelograms. The image of the unit square can be found by simply finding the images of the four corners of the square and constructing the parallelogram determined by those points. In this case, we have the figure. But there is more. $L$ is area preserving. Hence the image of the parallelogram will also have area 1. And due to the equivalence relation given by the exponential map on $\mathbb{R}^2$, every point in the image of the unit square has a representative within its equivalence class INSIDE the unit square. We can reconstruct the unit square by translating back all of these outside points back into the square. This becomes the image of points on the torus back into the torus.

See the drawing, where
\[
L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]
Question 18. How are any periodic points distributed?

we have the following proposition:

**Proposition 19.** The set of all periodic points of $F_L : \mathbb{T}^2 \to \mathbb{T}^2$ is dense in $\mathbb{T}^2$ and $P_n(F_L) = \lambda_1^n + \lambda_1^{-n} - 2$.

**Proof.** The first claim we will make to prove this result is the following: *Every rational point in $\mathbb{T}^2$ is periodic.* To see this, note that every rational point in $\mathbb{T}^2$ is a point in the unit square with coordinates $x = \frac{s}{q}$, and $y = \frac{t}{q}$, for some $q, s, t \in \mathbb{Z}$. For every point like this, $F_L(x, y)$ is also rational with the same denominator (neglecting simplification, do you see why?) But there are only $q^2$ distinct points in $\mathbb{T}^2$ which are rational and which have $q$ as the common denominator. Hence, at some point, $O(x, y)$ will start repeating itself. Hence this claim is proved. Now notice that the set of all rational points in $\mathbb{T}^2$ is dense in $\mathbb{T}^2$, or 

$$\overline{\mathbb{Q} \cap [0, 1]} \times \overline{\mathbb{Q} \cap [0, 1]} = [0, 1]^2.$$ 

Hence the periodic points are dense in $\mathbb{T}^2$.

The next claim is: *Only rational points are periodic.* To see this, assume $F_L \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix}$. Then 

$$F_L^n \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \mod 1,$$

and this forces the system of equations

$$ax + by = x + k,$$

$$cx + dy = y + \ell , \text{ for } k, l \in \mathbb{Z}.$$ 

Simply solve this system for $x$ and $y$ and you get that $x, y \in \mathbb{Q}$.

**Exercise 4.** Solve this system for $x$ and $y$.
The number of periodic points can be found by creating a new linear map. Define
\[ G_n \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = F^n_L \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) - \begin{bmatrix} x \\ y \end{bmatrix} = (F^n_L - I_2) \begin{bmatrix} x \\ y \end{bmatrix}. \]
The \( n \)-periodic points are precisely the kernel of this linear map:
\[ P_n(F_L) = \ker(G_n) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid G_n \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}. \]
We can easily count these now. They are precisely the pre-images of integer vectors!

**Claim.** All pre-images of \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) under the map \( G_n = F^n_L - I_2 \) are given by \( \mathbb{Z}^2 \cap (L^n - I_2)([0, 1] \times [0, 1]). \)

- Since \( F_L \) is to be understood as simply the matrix \( L \) where images are taken modulo 1, the map \( G_n \) is simply the map \( L^n - I_2 \) where images are taken modulo 1. Hence we can study the effect of \( G_n \) by looking at the image of \( L^n - I_2 \).
- To avoid over-counting points, we modify our unit square, eliminating twice-counted points (on the edges) and quadruply-counted points (the corners). Consider the “half-open box” \([0, 1)^2\) as our model of \( \mathbb{T}^2 \). In this model, every point lives in its own equivalence class.

We try a few early iterates:

**Example 20.** \( G_1 = L - I_2 \). Here
\[ G_1 = L - I_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \]
This map is a shear on \([0, 1)^2\) and we see that the only integer vector in the image is the origin. Thus
\[ P_1(F_L) = \lambda^1 + \lambda^{-1} - 2 = \frac{3 + \sqrt{5}}{2} + \frac{3 - \sqrt{5}}{2} - 2 = 3 - 2 = 1. \]

![Example 20 Diagram](image)

**Example 21.** \( G_2 = L^2 - I_2 \). Here
\[ G_2 = L^2 - I_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}. \]
This map is a little more complicated, and we see that there are a few more integer vectors in the image, namely the points $(2, 1)$, $(3, 2)$, $(4, 2)$, and $(5, 3)$. And since

$$P_2(F_L) = \lambda^2 + \lambda^{-2} - 2 = 7 - 2 = 5,$$

we see that the formula continues to hold (where is the fifth point?)

![Diagram of a map with points and vectors]

**Exercise 5.** What were the original points in $[0,1)^2$ that correspond to these 5 integer vectors under $G_2$?

This proof ends by establishing an interesting geometric link: The area of $G_n ([0,1)^2)$ is precisely equal the number of integer-vectors in the image. And the latter is given by

$$\det (G_n) = \lambda^n + \lambda^{-n} - 2,$$

where $\lambda$ is the largest eigenvalue (in magnitude) of $L$.

Note: $G_2$ on $\mathbb{R}^2$ is NOT area preserving! In fact,

$$\det(G_2) = \left| \begin{array}{cc} 4 & 3 \\ 3 & 1 \end{array} \right| = 5.$$

Note that the map $F_L$ above was area-preserving on the torus. It is also invertible (any determinant one matrix with integer coefficients is invertible, and the inverse is also of determinant one with integer entries!) However, area-preserving does NOT ensure invertibility of the map. The prime example is the circle map $E_m : S^1 \rightarrow S^1$, where $E_m(z) = z^m$. The map is area-preserving, if we sum all of the lengths of the disjoint pre-images of small sets. But it is also of degree $m$. And if $|m| > 1$, the map is $m$ to 1. Invertibility is a very desirable quality for a map, as it allows us to work both forwards and backwards in constructing orbits. Fortunately, there are ways to study non-invertible maps by encoding their information in a (different) invertible dynamical system. We will introduce this concept here, but not spend a lot of time on it for now.