1. Topological Conjugacy

**Definition 1.** Suppose $g : X \to X$ and $f : Y \to Y$ are maps of metric spaces and there exists a surjective map $h : X \to Y$ such that

$$h \circ g = f \circ h.$$ 

Then $f$ is called a factor of $g$ under $h$ and $f$ is said to be topologically semiconjugate to $g$ via the semiconjugacy $h$. Furthermore, if $h$ is a homeomorphism, then $h$ is a conjugacy and $f$ is topologically conjugate to $g$. We say in this case that $f \sim_h g$.

Note: In a (semi)conjugacy, orbits are taken to orbits via $h$. Thus the orbit structure of $g$ and that of $f$ are the same. It is for this that in dynamical systems, the notion of conjugacy is the notion of equivalence, or isomorphism. As we will see, the existence of a conjugacy allows us to study hard-to-study dynamical systems by instead establishing a conjugacy between them and easy to study ones.

The *tent map* $T_r : [0, 1] \to [0, 1]$ (at right) is a continuous, piece-wise linear, unimodular interval map given by

$$T_r(x) = \begin{cases} 
rx & \text{if } 0 \leq x \leq \frac{1}{2} \\
(1-x) & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}$$

This is also sometimes called the sawtooth function. Its height, at $x = \frac{1}{2}$, is obviously $\frac{r}{2}$.

In contrast, the linear expanding map $E_2$ on $S^1$ has the graph at left. As a map on $S^1$, it is certainly continuous (here, the point 0 is the same as 1 in both the domain and the range. Hence the map can run off the top of the graph and reappear at the bottom and still be continuous). As a graph in the unit square displays much of the same information as the tent map when the peak is precisely at 1. In fact, we can define $E_2$ as an interval map via

$$E_2(x) = \begin{cases} 
2x & \text{if } 0 \leq x \leq \frac{1}{2} \\
2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}$$

**Proposition 2.** The logistic map $f_4(x) = 4x(1-x)$ on $[0, 1]$ is topologically semi-conjugate to $E_2(x) = 2x \mod 1$ on $S^1$ via $h_1(x) = \sin^2 \pi x$, and topologically conjugate to the tent map $T_2 : [0, 1] \to [0, 1]$ via the conjugacy $h_2(x) = \sin^2 \frac{x}{2}$. 

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First, we will state without much detail that \( h_2(x) \) is a homeomorphism. It is continuous, 1-1 and onto \([0, 1]\), and its inverse \( h_2^{-1}(x) = \frac{2}{\pi} \arcsin \sqrt{x} \) is also continuous on \([0, 1]\). Instead, the map \( h_1 : S^1 \to I \) is a 2-1 map, and hence does not have an inverse. Thus \( h_1(x) \) is not a homeomorphism (which makes sense, since \( S^1 \) is not homeomorphic to \( I \)). \( h_1(x) \) is surjective, however, as one can see here.

**Proof.** Here, we will explicitly show the conjugacies. First, we show \( h_1 \circ E_2 = f_4 \circ h_1 \). This semi-conjugacy condition needs to be parsed along the linear pieces of \( E_2 \). Hence we want

\[
(1) \quad h_1(2x) = f_4 (\sin^2 \pi x) \quad \text{for} \quad 0 \leq x \leq \frac{1}{2}
\]

\[
(2) \quad h_1(2x - 1) = f_4 (\sin^2 \pi x) \quad \text{for} \quad \frac{1}{2} \leq x \leq 1.
\]

As for the left hand sides of these two equations, in Equation (1), we get

\[
h_1(2x) = \sin 2 \pi (2x) = \sin^2 2\pi x.
\]

On the right hand side of each, we see

\[
f_4 (\sin^2 \pi x) = 4 (\sin^2 \pi x) (1 - \sin^2 \pi x)
\]

\[
= 4 (\sin^2 \pi x) (\cos^2 \pi x)
\]

\[
= 4 \left( \frac{1}{2} - \frac{1}{2} \cos 2\pi x \right) \left( \frac{1}{2} + \frac{1}{2} \cos 2\pi x \right)
\]

\[
= 4 \left( \frac{1}{4} - \frac{1}{4} \cos^2 2\pi x \right)
\]

\[
= 4 \left( \frac{1}{4} \sin^2 2\pi x \right) = \sin^2 2\pi x.
\]

As for the conjugacy \( h_2(x) \), we need to show that \( h_2 \circ T_2 = f_4 \circ h_2 \). Again, we would need to parse this condition along the two linear pieces of \( T_2 \). The two resulting equations are almost identical to the previous case. In fact, Equation (2) is precisely the same with all of the factors \( \pi \) replaced by \( \frac{\pi}{2} \) (thereby replacing \( h_1 \) with \( h_2 \)). And for Equation (1), this time we get

\[
h_2(2 - 2x) = \sin^2 \frac{\pi}{2} (2 - 2x) = \sin^2 \pi (1 - x) = \sin^2 \pi - \pi x = \sin^2 2\pi x
\]

since \( \sin(\pi - x) = \sin x \).

\[\square\]

**Notes:**
• The maps $h_1$ and $h_2$ are truly related, and come from the relationship between $S^1$ and $I$. Conjugacies are really all about maps that take orbits to orbits, and any map that satisfies this condition will transfer the dynamics of one system to the other. In this case, both the tent map and the expanding circle map have a certain symmetry about them; $E_2\left(x + \frac{1}{2}\right) = E_2(x)$ on $I$, while $T_2(x) = T_2(1 - x)$. $f_4$ shares the latter property with $T_2$, and $T_2(x) = 1 - E_2(x)$ on the interval $\frac{1}{2} \leq x \leq 1$. The sine function has the appropriate property that $\sin \pi x = \sin \pi (1 - x)$. The sine function is also a beautiful way to map $S^1$ down onto an interval. Indeed, view points of $S^1$ as $e^{2\pi ix}$, for $x \in I$, and the real part of $z = e^{2\pi ix} \in S^1$ is $\cos 2\pi x$. We can scale this as a “tent-like” map on $I$ as the function

$$x \mapsto \frac{1 - \cos 2\pi x}{2} = \frac{1}{2} - \frac{1}{2} \cos 2\pi x = \sin^2 \pi x.$$  

This is precisely $h_1$ above. For $h_2$, halving the angle makes $h_2$ 1-1 on $I$.

• Once a conjugacy is specified, ALL of the interesting dynamics of the logistic map for $\lambda = 4$ are present in the tent map for $r = 2$. This is not quite true for a semiconjugacy.

2. Topological Entropy

Recall our indicators of dynamical complexity from before: topological transitivity, minimality, density of periodic orbits, chaos, growth rates of periodic orbits, etc. These properties are called dynamical invariants under conjugacy:

• Allows you to study a new dynamical system by establishing a conjugacy with a known one.
• Allows you to classify dynamical systems (everything conjugate to a known dynamical system has the same dynamical invariants).

Theorem 3. Any two degree-2, expanding maps of $S^1$ are conjugate.

Thus, the only degree-2 expanding map to study is the map $E_2 : S^1 \to S^1$, $E_2(x) = 2x \mod 1$.

Example 4. Rotation number classifies circle homeomorphisms

Here is a new dynamical invariant:

Definition 5. The topological entropy of a map is the exponential growth rate of the number of orbit segments distinguishable with arbitrary precision,

Note: This is the discrete version of what are called Lyapunov exponents.

Definition 6. Lyapunov Exponents are numbers which represent the exponential rate of divergence of nearby trajectories.
The idea here is to take two trajectories of initial separation $\delta_0 > 0$. If after time-$t$, the separation is $\delta_t = e^{\lambda t} \delta_0$, then the Lyapunov exponent is $\lambda$. Note that this depends largely on the direction of the measurement. Different directions of travel will result in different separation rates. Of interest typically is the largest:

- For $C^1$-dynamical systems, the exponents are related directly to the eigenvalues of the Jacobian matrix, a local linearization of the system.
- For $C^0$-systems, there is no Jacobian matrix to work with. However, one can still calculate the maximum exponent via
  \[
  \lambda = \lim_{t \to \infty} \frac{1}{t} \lim_{\delta_0 \to 0} \log \frac{\delta_t}{\delta_0}.
  \]
- Calculations of Lyapunov exponents are usually done numerically and only locally. Only rarely can they be calculated analytically or over the entire space.

Now to actually define topological entropy, we will need some more machinery:

2.1. **Capacity.** Let $X$ be a compact metric space (both closed and bounded). A set $E \subset X$ is called $r$-dense if, using the metric,

\[ X \subset \bigcup_{x \in E} B_r(x). \]

That is, if $X$ can be covered by a set of $r$-balls all of whose centers lie in $E$. Then the $r$-capacity of $X$, with metric $d$ is the minimal cardinality of any $r$-dense set. Denote the $r$-capacity of a set $X$ by $S_{X,d}(r)$ (or simply $S_d(r)$ when the space $X$ is either understood or not necessary to be explicit about).

Some Notes:

- This is simply a way of denoting the “thickness” of sets which have no actual volume by how they sit inside $X$ (think cantor sets sitting inside an interval).
- It does not really matter ultimately, but we will mostly consider closed balls in these calculations.
- Some examples:

  **Example 7.** $\mathbb{Z}$ is $r$-dense in $\mathbb{R}$ if $r > \frac{1}{2}$ if the balls are open, and $r \geq \frac{1}{2}$ is the balls are closed.

  **Example 8.** $\mathbb{Z}^2$ is $r$-dense in $\mathbb{R}^2$ if $r > \frac{\sqrt{2}}{2}$ if the balls are open, and $r \geq \frac{\sqrt{2}}{2}$ is the balls are closed. Can you visualize this?

  **Example 9.** Let $I = [0,1]$ be the unit interval. Using open balls here, the $\frac{1}{2}$-capacity of $I$ is 3. The $\frac{1}{r}$-capacity is 5. One can show that $S_d\left(\frac{1}{2^n}\right) = 2^n + 1$, and more generally $S_d(r) = \left[\frac{1}{r}\right] + 1$. 

Exercise 1. Show this.

Exercise 2. Determine a bound on \( r \) for which \( \mathbb{Z}^3 \) is \( r \)-dense in \( \mathbb{R}^3 \).

- These calculations work well with Cantor Sets. Studying how \( S_d(r) \) changes as \( r \) changes (really, it is the order of magnitude of \( S_d(r) \)) leads to a generalized notion of dimension.

3. Box Dimension

A rough notion of dimension for a topological space would be how many coordinates it would take to completely determine a point in the space (in relation to the other points). For example, the common description of the two-sphere \( S^2 \) is as the unit-sphere in \( \mathbb{R}^3 \); the set of all unit-length vectors in \( \mathbb{R}^3 \). However, using spherical coordinates \((\rho, \theta, \phi)\) (see the connection), all of these points have coordinate \( \rho = 1 \), and hence each point on the sphere only requires two coordinates to differentiate between them. Hence, in a way, \( S^2 \) is two-dimensional as a space. This notion is not mathematically precise, however, as there do exist curves (1-dimensional lines) that can “fill” a two-dimensional space (Peano curves, some examples are called). Hence is this curve 1-dimensional, or 2 dimensional? Here, we will explore one mathematically precise notion of dimension (there are many), which will be useful in our definition of topological entropy.

Definition 10. A metric space \( X \) is called totally bounded if \( \forall r > 0, X \) can be covered by a finite set of \( r \)-balls all of whose centers are in \( X \).

Really, this definition is technical, and is meant to account for the general metric space aspect of this discussion. That the centers need to be within \( X \) really only is a factor when the metric space \( X \) is a subspace of another space \( Y \) (otherwise there is no “outside” of \( X \). And in Euclidean space, the notion of totally bounded is just the common notion of bounded that you are used to.

Definition 11. For \( X \) totally bounded,

\[
\text{bdim}(X) := \lim_{r \to 0} \frac{-\log S_{(X,d)}(r)}{\log r}
\]

is called the box dimension of \( X \).

Notes:

- This concept is also called the Minkowski-Bouligard dimension, or the entropy dimension or the Kolmogorov dimension.
- This is an example of the idea of fractional dimension; some sets may look bigger then 0-dimensional, yet smaller than 1-dimensional, for example.
In the case where this limit may not exist (I cannot think of an example where it wouldn’t for a totally bounded set), certainly one can use the limit superior or the limit inferior to gain insight as to the “size” of a set.

To calculate, really simply find a sequence of $r$-sizes going to 0, and calculate the $r$-capacities for this sequence. If the limit exists, then ANY sequence of $r$’s going to 0, with their associated $r$-capacities will determine the same box dimension (Why?).

**Example 12.** Calculate $\text{bdim}(I)$, for $I[0, 1]$ with the metric $d$ that $I$ inherits from $\mathbb{R}$. Recall that if we were to use closed balls, then the $\frac{1}{2^n}$-capacity for $I$ is $S_{(X,d)}(\frac{1}{2^n}) = 2^{n-1}$. But for open balls, we have $S_{(X,d)}(\frac{1}{2^n}) = 2^{n-1} + 1$. The box dimension should be the same for both. Indeed, it is: For the harder one,

$$\text{bdim} (I) = \lim_{r \to 0} \frac{-\log S_{(X,d)}(r)}{\log r} = \lim_{n \to \infty} \frac{-\log (2^{n-1} + 1)}{\log \left(\frac{1}{2^n}\right)} = \lim_{n \to \infty} \frac{\log (2^{n-1} + 1)}{\log 2^n}$$

$$\geq \lim_{n \to \infty} \frac{\log 2^{n-1}}{\log 2^n} = \lim_{n \to \infty} \frac{n-1}{n} = 1,$$

and

$$\text{bdim} (I) = \lim_{r \to 0} \frac{-\log S_{(X,d)}(r)}{\log r} = \lim_{n \to \infty} \frac{-\log (2^{n-1} + 1)}{\log \left(\frac{1}{2^n}\right)} = \lim_{n \to \infty} \frac{\log (2^{n-1} + 1)}{\log 2^n}$$

$$\leq \lim_{n \to \infty} \frac{\log 2^{n-1}}{\log 2^n} = \lim_{n \to \infty} \frac{\log 2^{n-1}}{\log 2^n} + \lim_{n \to \infty} \frac{\log n}{\log 2^n}$$

$$= \lim_{n \to \infty} \frac{n-1}{n} + \lim_{n \to \infty} \frac{\log n}{n} = 1 + 0 = 1.$$

Hence $\text{bdim} (I) = 1$. Using the closed ball construction is even easier.

**Example 13.** Let $C$ be the Ternary Cantor Set. Show $\text{bdim}(C) = \frac{\log 2}{\log 3}$. Here, assume that $C$ sits inside $I$ from the previous example, and again inherits its metric $d$ from $I$. And since we can choose our sequence of $r$’s going to zero, we will choose $r = \frac{1}{3^n}$, and consider only closed balls. Then one can show that $S_{(C,d)}(\frac{1}{3^n}) = 2^{n+1}$. (Think about this: At each stage, we remove the middle third of the remaining intervals. That means that at each stage we can cover each interval by a closed ball of radius $\frac{1}{3^n}$. But the mid-point is NOT in $C$. Hence we have to shift over a bit to find a point in $C$. Which means that we will need another ball to cover the remainder on this side. This over covers the interval, but is not enough to cover two adjacent intervals. And since at each stage there are $2^{n+1}$ intervals, we are done. See the figure.
The calculation is now easy:

\[
\text{bdim}(C) = \lim_{r \to 0} \frac{-\log S_{(C,d)}(r)}{\log r} = \lim_{n \to \infty} \frac{-\log (2^{n+1})}{\log \left(\frac{1}{3^n}\right)} = \lim_{n \to \infty} \frac{\log (2^{n+1})}{\log 3^n}
\]

\[
= \lim_{n \to \infty} \frac{n + 1}{n} \cdot \frac{\log 2}{\log 3} = \frac{\log 2}{\log 3}.
\]

**Exercise 3.** Let \( B = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\} \). Calculate \( \text{bdim}(B) \).

In fact, we have the following:

**Theorem 14.** Let \( C \subset I \) be the Cantor set formed by removing the middle interval of relative length \( 1 - \frac{2}{\alpha} \) at each stage. Then

\[
\text{bdim}(C) = \frac{\log 2}{\log \alpha}.
\]

A special note: All Cantor sets are homeomorphic. Yet, if we change the size of a removed interval at each stage, we effectively change the box dimension. This means that box dimension is NOT a topological invariant (remains the same under topological equivalence). Since a homeomorphism here would also act as a conjugacy between two dynamical systems on Cantor Sets, this also means that box dimension is also NOT a dynamical invariant.

For \( f : X \to X \), a continuous map on a metric space \((X, d)\), consider a sequence of new metrics on \( X \) indexed by \( n \in \mathbb{N} \):

\[
d^f_n(x, y) := \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)).
\]

Here, the new metrics \( d^f_n \) actually measure a “distance” between orbit segments

\[
\mathcal{O}_{x,n} = \{x, f(x), \ldots, f^{n-1}(x)\},
\]

\[
\mathcal{O}_{y,n} = \{y, f(y), \ldots, f^{n-1}(y)\}
\]

as the farthest that these two sets diverge along the orbit segment, and assigns this distance to the pair \( x \) and \( y \).

**Exercise 4.** Show for a given \( n \) that \( d^f_n \) actually defines a metric on \( X \).

Now, using the metric \( d^f_n \), we can define an \( r \)-ball as the set of all neighbor points \( y \) whose \( n \)th orbit-segment \( \mathcal{O}_{y,n} \) stays within \( r \) distance of \( \mathcal{O}_{x,n} \):

\[
B_r(x, n, f) = \left\{ y \in X \left| d^f_n(x, y) < r \right. \right\}.
\]

Convince yourself that as we increase \( n \), the orbit segment is getting longer, and more and more neighbors \( y \) will have orbit segments that move away from \( \mathcal{O}_{x,n} \). Thus the \( r \)-ball will
get smaller as \( n \) increases. But by continuity, the \( r \)-balls for any \( n \) will always be open sets in \( X \) that have \( x \) as an interior point. Also, as \( r \) goes to 0, the \( r \)-balls will also get smaller, right?

Now define the \( r \)-capacity of \( X \), using the metric \( d^f_n \) and the new \( r \)-balls \( B_r(x, n, f) \), denoted \( S_{(X, d^f)^n}(r, n, f) \) (this is the SAME notion of \( r \)-capacity as the one we used for the box dimension! We are only changing the metric on \( X \) to \( d^f_n \)). But the actual calculations of the \( r \)-capacity depend on the choice of metric). As before, as \( r \) goes to 0, the \( r \)-balls shrink, and hence the \( r \)-capacity grows. And also, as \( n \) goes to \( \infty \), we use the different \( d^f_n \) to measure ultimately the distances between entire positive orbits. This also forces the \( r \)-balls to shrink, and hence the \( r \)-capacity to grow. What is the exponential growth rate of the \( r \)-capacity as \( r \to 0 \)? This is the notion of topological entropy:

**Definition 15.** Let

\[
\begin{align*}
  h_d(f, r) &:= \lim_{n \to \infty} \frac{\log S_d(r, n, f)}{n} \\
  h_d(f) &:= \lim_{r \to 0} h_d(f, r)
\end{align*}
\]

is called the *topological entropy* of the map \( f \) on \( X \).

There are many things to say about this. To start:

- Topological entropy is a measure of the tendency of orbits to diverge from each other. It will always be a non-negative number, and the higher it is, the faster orbits are diverging. In Euclidean space, maybe this is not so special (think of the linear map on \( \mathbb{R}^2 \) given by the matrix \( \lambda I_2 \), with \( \lambda > 1 \). All orbits diverge, but the dynamics is not very interesting), but in a compact space with all orbits diverging, the resulting messy nature of the dynamics can be quite interesting. Thus, topological entropy is a measure of the orbit complexity, and the higher the number, the more interesting (read messy) the dynamical structure.

- Another common notation for topological entropy is \( h_{top}(f) \) or \( h_T(f) \) or even \( h(f) \). These are, in a sense, more accurate since it turns out that the topological entropy of a map does not actually depend on the metric \( d \), at least up to equivalence, chosen for use in its definition. It is possible, however, that inequivalent metrics may lead to either the same or a different entropy. We will use the notation \( h(f) \) in our subsequent discussion.

- Contractions and isometries have no entropy:

  **Proposition 16.** Let \( f \) be either a contraction or an isometry. Then \( h(f) = 0 \).

  *Proof.* In the case of \( f \) an isometry, for any \( n \in \mathbb{N} \), \( d^f_n = d \), since distances between iterates of a map are the same as the original distances between the initial points. Hence the \( r \)-capacity \( S_{(X, d)^n}(r, n, f) = S_{(X, d)}(r, f) \) does not depend on \( n \), and hence \( h(r, f) = 0 \). For a contraction, the iterates of two distinct points are always closer
together than the original points. Hence also here $d_n^f = d$. This leads to the same conclusion.

- Topological entropy is a dynamical invariant (invariant under conjugacy). This means that if $f$ is conjugate to $g$, then $h(f) = h(g)$. However, it is also useful to use the contrapositive: If one has two maps where $h(f) \neq h(g)$, then it is not possible that $f$ is conjugate to $g$.
- Topological entropy measures, in a way, the exponential growth rate of the number of trajectories that are $r$-separable after $n$ iterations. Suppose this number is proportional to $e^{nh}$. Then $h$ would be the growth rate for a fixed $r$, and as $r \to 0$, this $h$ would tend to the entropy.
- Defining the topological entropy for a flow is simply a matter of replacing the $n \in \mathbb{N}$ with $t \in \mathbb{R}$ in all of the definitions for the invariant. We can relate the two in a way: The topological entropy of a flow is equal to the topological entropy of its time-1 map (really, its time-$t$ for any choice of $t$, since the flow provides the conjugacy of any $t$-map with any other).
- In practice, topological entropy is quite hard to calculate. However, in many cases, and in response to the last bullet point, the entropy is directly related to the largest Lyapunov exponent of the system, at least for $C^1$ systems.