Subspaces
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I did not get to make it to subspaces today in class, so I decided to make this study sheet for you guys to briefly discuss Sub Spaces.

1 Introduction
We all know what Vector Spaces are (ie. \( \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \), etc) and we also know that they have many properties. A few of the most important are that Vector Spaces are closed both under addition and scalar multiplication. What does that mean? Being closed under addition means that if we took any vectors \( x_1 \) and \( x_2 \) and added them together, their sum would also be in that vector space.

ex. Take \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) and \( \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \). Both vectors belong to \( \mathbb{R}^3 \). Their sum, which is \( \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \) is also a member of \( \mathbb{R}^3 \).

Being closed under scalar multiplication means that vectors in a vector space, when multiplied by a scalar (any real number), it still belongs to the same vector space.

ex. Consider \( \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \). If I multiply the vector by a scalar, say, 10, I will get \( 10 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 40 \\ 30 \end{pmatrix} \). Which is still in \( \mathbb{R}^3 \)

There are two very important notions of a Vector Space, and will end up being very important in defining a Sub Space.

2 Subspaces
Now we are ready to define what a subspace is. Strictly speaking, A Subspace is a Vector Space included in another larger Vector Space. Therefore, all properties of a Vector Space, such as being closed under addition and scalar multiplication still hold true when applied to the Subspace.

ex. We all know \( \mathbb{R}^3 \) is a Vector Space. It satisfies all the properties including being closed under addition and scalar multiplication. Consider the set of all vectors \( S = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \) such at x and y are real numbers. This is also a Vector Space because all the conditions of a Vector Space are satisfied, including the important conditions of being closed under addition and scalar multiplication.

ex. Consider the vector \( \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \) Which are both contained in \( S \). If we add them together we get \( \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix} \), which is is still in \( S \). We can also multiply each one by a scalar, say \( \frac{1}{2} \) and get \( \begin{pmatrix} \frac{1}{2} \\ 2 \\ 0 \end{pmatrix} \), and \( \begin{pmatrix} \frac{5}{2} \\ 1 \\ 0 \end{pmatrix} \), which are both
So we see that $S$ is a Vector Space, but it is important to notice that all of $S$ is contained in $\mathbb{R}^3$. By this, I mean any vector in $S$ can also be found in $\mathbb{R}^3$. Therefore, $S$ is a SUBSPACE of $\mathbb{R}^3$.

Other examples of Sub Spaces:

- The line defined by the equation $y = 2x$, also defined by the vector definition $\begin{pmatrix} t \\ 2t \end{pmatrix}$ is a subspace of $\mathbb{R}^2$

- The plane $z = -2x$, otherwise known as $\begin{pmatrix} t \\ 0 \\ -2t \end{pmatrix}$ is a subspace of $\mathbb{R}^3$

- In fact, in general, the plane $ax + by + cz = 0$ is a subspace of $\mathbb{R}^3$ if $abc \neq 0$. This one is tricky, try it out. Test whether or not any arbitrary vectors $x_1$, and $x_s$ are closed under addition and scalar multiplication.

### 2.1 Subspace Test

Given a space, and asked whether or not it is a Sub Space of another Vector Space, there is a very simple test you can preform to answer this question. There are only two things to show:

**The Subspace Test** To test whether or not $S$ is a subspace of some Vector Space $\mathbb{R}^n$ you must check two things:

1. if $s_1$ and $s_2$ are vectors in $S$, their sum must also be in $S$
2. if $s$ is a vector in $S$ and $k$ is a scalar, $ks$ must also be in $S$

In other words, to test if a set is a subspace of a Vector Space, you only need to check if it closed under addition and scalar multiplication. Easy!

ex.

Test whether or not the plane $2x + 4y + 3z = 0$ is a subspace of $\mathbb{R}^3$.

To test if the plane is a subspace, we will take arbitrary points $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$, and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, both of which lie on the plane, and we will check both points of the subspace test.

1. Closed under addition: Consider $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$. We will test if the point also lies in the plane. We will take our original polynomial, $2x + 4y + 3z = 0$, and substitute $x$ with $x_1 + x_2$, $y$ with $y_1 + y_2$, and $z$ with $z_1 + z_2$ and get

$$2(x_1 + x_2) + 4(y_1 + y_2) + 3(z_1 + z_2) = 0$$

From here we can distribute and get:

$$2x_1 + 2x_2 + 4y_1 + 4y_2 + 3z_1 + 3z_2 = 0$$

which we can reorganize to get

$$(2x_1 + 4y_1 + 3z_1) + (2x_2 + 4y_2 + 3z_2) = 0$$

We also know that both $2x_1 + 4y_1 + xz_1$ and $2x_2 + 4y_2 + 3z_2$ are both 0, so the equation becomes $0 + 0 = 0$, which proves that the plane is closed under addition.
2. Closed under scalar multiplication: Consider the point \( \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \) on the plane. Also consider the scalar \( k \). If we multiply \( k \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} kx_1 \\ ky_1 \\ kz_1 \end{pmatrix} \) we need to check if this is also in the plane. To do this, we will plug in the point into the original plane. So we have

\[
2kx_1 + 4ky_1 + 3kz_1 = 0
\]

which factors into

\[
k(2x_1 + 4y_1 + 3z_1) = 0
\]

And because we know \( 2x_1 + 4y_1 + 3z_1 = 0 \) we obtain \( 0 = 0 \). So the plane is closed under scalar multiplication.

Phew! We proved both parts of the Subspace test so we have proved that the plane defined by the equation \( 2x_1 + 4y_1 + 3z_1 = 0 \) is a subspace of \( \mathbb{R}^3 \).