Research statement

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1 Background and motivations: discrete subgroups and lattices in $PU(2,1)$

The setting of my research is the still rather unexplored area of discrete groups of isometries of complex hyperbolic space, in particular that of lattices in $PU(2,1)$. This space is one of the two occurrences (with real hyperbolic space) of a rank 1 symmetric space in which Margulis' super-rigidity result does not apply. But, as opposed to the real hyperbolic case, very little is known about these groups, beginning with interesting examples. Recall that there exist in the real case reflection groups (see the constructions of Vinberg, Makarov in small dimensions), and that there are examples of non-arithmetic lattices in all dimensions, by the construction of Gromov–Piatetski-Shapiro.

In the case of $SU(n,1)$, all these questions remain wide open, even in the smallest dimensions. This is mostly due to the fact that there are no totally geodesic real hypersurfaces in complex hyperbolic geometry, and in particular no natural notion of polyhedra or reflection groups as in the real hyperbolic (or Euclidean) case. This situation makes it difficult to construct not only discrete subgroups of $SU(n,1)$, but also fundamental polyhedra for such groups. Possible substitutes in $H^n_{\mathbb{C}}$ are *complex reflections* (or $\mathbb{C}$-reflections) which are holomorphic isometries fixing pointwise a totally geodesic complex hypersurface (a copy of $H^{n-1}_{\mathbb{C}} \subset H^n_{\mathbb{C}}$), and *real reflections* (or $\mathbb{R}$-reflections) which are antiholomorphic involutions fixing pointwise a Lagrangian subspace (a copy of $H^n_{\mathbb{R}} \subset H^n_{\mathbb{C}}$). Most examples up to now are generated by complex reflections, or are arithmetic constructions. The use of $\mathbb{R}$-reflections and of the corresponding Lagrangian planes is very recent (it was introduced by Falbel and Zocca in [FZ] in 1999) and is one of the guiding principles of my research.

The first constructions of lattices in $SU(2,1)$ go back to the end of the XIXth century and are due to Picard; it is striking that they are still almost the only known examples. Picard's first examples are of an arithmetic nature, in the spirit of the Bianchi groups; namely he considered in [Pic1] the groups $SU(2,1,\mathcal{O}_d) \subset SU(2,1)$ comprising the matrices whose entries are all in the imaginary quadratic ring of integers $\mathcal{O}_d$ (d is square-free and negative), such as the Gaussian integers $\mathbb{Z}[i]$ or the Eisenstein integers $\mathbb{Z}[e^{2i\pi/3}]$.

More surprising is Picard's second construction coming from monodromy groups of the so-called *hypergeometric* functions, which are meromorphic functions of a complex variable (see [Pic2] and [DM] for a more modern language), which produces a subgroup of $PU(2,1)$ generated by three complex reflections. We now consider $PU(2,1)$ (a quotient of order 3 of $SU(2,1)$) which is more natural geometrically, as the group of holomorphic isometries of $H^n_{\mathbb{C}}$. This construction is related to the moduli space of quintuples of points on the Riemann sphere $\mathbb{C}P^1$, as well as to Euclidean cone metrics on the sphere studied by Thurston in [Th] (see also [Par1]). Picard stated conditions sufficient to ensure dicreteness of the corresponding group (conditions which he obtained through a geometric analysis). The 27 corresponding lattices are now known as the *Picard lattices*; among them, 7 are nonarithmetic. This observation is due to Deligne and Mostow (the notion was not even burgeoning in Picard's time), when they rewrote and made systematic Picard's hypergeometric approach a century later in [DM]. They obtained by this method lattices in $PU(n,1)$ for $n \leqslant 5$; it is interesting to note that in dimension 2 they find exactly the 27 Picard lattices (although using a weaker discreteness condition) and that the only other nonarithmetic lattice of their list is a non-cocompact example in $PU(3,1)$.

In the meantime, at the end of the 1970's, Mostow had studied new examples of subgroups of $PU(2,1)$ generated by three complex reflections of order p ($p = 3,4,5$), which he denoted by $\Gamma(p,t)$ (t being a real parameter). His investigation [Mos1] is based on a detailed analysis of the geometric action of the group on $H^2_{\mathbb{C}}$, from which he infers discreteness conditions as well as the description of a fundamental domain when this is the case. These examples, still poorly understood, were the other guiding principle of my thesis. Mostow's original approach is the construction of a fundamental domain by Dirichlet's method, which consists in the consideration, for a "central" point p_0, of the set of points closer to p_0 than to any other point in its orbit. This approach has the advantages of generality and simplicity of principle, but it is very hard to use in practice. Namely, the basic object in this construction is the set of points equidistant from two given points (p_0 and one of its translates), which Mostow called a *bisector*. These objects enjoy remarkable geometric properties, such as the double foliation by $\mathbb{C}$-planes and $\mathbb{R}$-planes (see [Gol]), but their intersections are difficult to understand. For instance, one of the goals of Goldman's book [Gol] is to understand these intersections; see also the third chapter of this thesis (in particular its introduction) for problems related to this construction.

2 Results of the thesis

One of the major problems for the construction of polyhedra in $H^2_{\mathbb{C}}$ is the absence of hyperplanes (or totally geodesic real hypersurfaces), which have no natural substitute. Different constructions of hypersurfaces have appeared, the nature of which is linked to the type of the group in question. Schwartz has described some different examples (in 2002) in his survey [Sz]; an idea shared by all these constructions is the use of $\mathbb{C}$-planes and $\mathbb{R}$-planes (and their boundaries, the $\mathbb{C}$-circles and $\mathbb{R}$-circles of the Heisenberg group $H^3 = \partial H^2_{\mathbb{C}}$) to foliate the hypersurfaces as in the case of bisectors.

This is where the $\mathbb{R}$-reflections which appear in the group play a crucial role, according to the general principle by which the most natural fundamental domain for a group rests upon the fixed-point loci of certain elements of the group. We thus gain some geometric information by decomposing the generators of a group of isometries as products of $\mathbb{R}$-reflections, which is a recurring theme in this thesis. A classic example of this situation is that of triangle groups of the plane (hyperbolic, Euclidean, or spherical), generated by two rotations which one decomposes as a product of reflections to determine the order of the product (and the third angle of the triangle bounded by the reflecting lines, which is a fundamental domain for the group). Another example arises in our investigation of Mostow's lattices in chapter 3 (see [DFP]), where we notice that a group slightly bigger than his lattice is generated by two isometries which decompose as a product of three $\mathbb{R}$-reflections. The corresponding $\mathbb{R}$-planes could be seen by transparency in Mostow's article (typically, certain quintuples of vertices of his domain were contained in one of these $\mathbb{R}$-planes), whithout his noticing them (see for this aspect figure 14.2 on p. 239 of [Mos1]). This allowed us to simplify the structure of the fundamental polyhedron by introducing some 2-faces contained in these $\mathbb{R}$-planes.

My thesis is organized as follows. After a short chapter of geometric preliminaries in $H^2_{\mathbb{C}}$ (for which the reader may also refer to Goldman's book [Gol]), we investigate the simplest kind of discrete groups, namely finite groups. In our setting, these groups are conjugate to subgroups of $U(2)$; it is thus a linear question in $\mathbb{C}^2$, but which already contains some rich geometric aspects. One of the motivations for this study was to understand the elementary building block of Mostow's lattices, the finite groups generated by two of the three fundamental complex reflections (this is the example of the group 3[3]3 in Coxeter's notation, which we study in detail). We describe precisely the role of $\mathbb{R}$-planes in that setting, and construct fundamental domains in the boundary of the unit ball of $\mathbb{C}^2$ based upon arcs of the corresponding $\mathbb{R}$-circles. We detail the conditions under which a given $\mathbb{R}$-reflection decomposes an elliptic isometry of $H^2_{\mathbb{C}}$ (in the sense where the latter can be written as the product of the given $\mathbb{R}$-reflection with another $\mathbb{R}$-reflection), and infer among other things the following result:

Theorem 2.1 *Every finite subgroup of $U(2)$ is of index 2 in a group generated by $\mathbb{R}$-reflections. More precisely:*

- *Every two-generator subgroup of $U(2)$ is of index 2 in a group generated by 3 $\mathbb{R}$-reflections.*
- *The exceptional finite subgroups of $U(2)$ which are not generated by two elements are of index 2 in a group generated by 4 $\mathbb{R}$-reflections.*

This second chapter is a joint work with E. Falbel and has been published in *Geometriae Dedicata* ([FPau]).

The next chapter consists in a detailed analysis of Mostow's lattices $\Gamma(p,t)$, based on the aforementioned observation that these lattices naturally contain $\mathbb{R}$-reflections. We construct a new fundamental domain Π which is simpler than Mostow's, but mostly which allows the use of synthetic geometric arguments which spare the resort to massive computer use for the proofs as in [Mos1]. Further motivations and ramifications can be found in the detailed introduction to that chapter, as well as some notation. $\tilde{\Gamma}(p,t)$ denotes the group generated by one of the complex reflections, say R_1 and the isometry J which cyclically permutes these three reflections; it contains $\Gamma(p,t)$ with index 1 or 3. We can sum up our results in the following:

Theorem 2.2 *The group $\tilde{\Gamma}(p,t) \subset PU(2,1)$, for $p = 3, 4,$ or 5 and $|t| < \frac{1}{2} - \frac{1}{p}$, is discrete if $k = (\frac{1}{4} - \frac{1}{2p} + \frac{t}{2})^{-1}$ and $l = (\frac{1}{4} - \frac{1}{2p} - \frac{t}{2})^{-1}$ are in $\mathbb{Z}$. In that case Π is a fundamental domain with side pairings given by J, R_1, R_2, R_2R_1, R_1R_2 and the cycle relations give the following presentation of the group*

$$\tilde{\Gamma}(p,t) = \langle J, R_1, R_2 \mid J^3 = R_1^p = R_2^p = J^{-1}R_2JR_1^{-1} = R_1R_2R_1R_2^{-1}R_1^{-1}R_2^{-1}$$
$$= (R_2R_1J)^k = ((R_1R_2)^{-1}J)^l = I\rangle.$$

This third chapter is a joint work with M. Deraux and E. Falbel and has been published in *Acta Mathematica* ([DFP]).

The last chapter is a first step in the search for new discrete groups in a family which we will call *elliptic triangle groups*. These are the groups generated by two elliptic isometries A and B (i.e. each having a fixed point inside $H^2_{\mathbb{C}}$) whose product AB is also elliptic. In the same way that we use the characterization of triangles of the plane (hyperbolic, Euclidian, or spherical) to understand the groups generated by two rotations, we will need the characterization of which conjugacy classes the product AB can be in when A and B are each in a fixed conjugacy class.

It is a classical problem in a linear group to characterize the possible eigenvalues of matrices $A_1, ..., A_n$ satisfying $A_1...A_n = 1$. In the group $GL(n,\mathbb{C})$, this question, known as the Deligne-Simpson problem, has arisen from the study of so-called Fuchsian differential systems on Riemann's sphere $\mathbb{C}P^1$ and is closely related to the Riemann-Hilbert problem (Hilbert's 21^{st} problem); see [Ko] for a survey of these questions and the partial answers which are known so far. The antipodal case of the compact group $U(n)$ has also been extensively studied and essentially solved in [AW], [Be], [Bi], [Kl1]; it is related to many surprising branches of mathematics as is pointed out in the surveys [F] and [Kl2]. The case of $U(n)$ is also studied in relation with Lagrangian subspaces and reflections of $\mathbb{C}^n$ in [FW1], from which we have adapted some ideas to the setting of the non-compact group $PU(2,1)$.

Elliptic conjugacy classes in $PU(2,1)$ are characterized by an unordered angle pair, so that the question is the determination of the image in the surface $\mathbb{T}^2/\mathfrak{S}_2$ of the map $\tilde{\mu}$, which is the composite of the group product (restricted to the product of fixed conjugacy classes), followed by projection from the group to its conjugacy classes. This is an occurrence of a momentum map associated with a quasi-Hamiltonian group action on a symplectic manifold, generalizing the classical Hamiltonian setting and defined in [AMM].

We describe the image of this momentum map $\tilde{\mu}$, and answer the related question of classifying triples of pairwise intersecting Lagrangian subspaces of $H^2_{\mathbb{C}}$ (or $\mathbb{R}$-planes). The image turns out to be the union of at most three (possibly overlapping) convex polygons in the surface $\mathbb{T}^2/\mathfrak{S}_2$. In particular it is not always convex (and not even locally convex when the different polygons overlap).

In the classical case there are general theorems which ensure that the image is convex, the most famous of which is the Atiyah-Guillemin-Sternberg theorem (see [A], [GS1], [GS2]), which states, in the case of a torus action on a compact connected symplectic manifold, that the image is a convex polytope, the convex hull of the image of fixed points under the group action. In the case of non-compact group actions, some conditions on the target are required (see [W1]) to obtain such a convexity theorem.

Our description of the image can be summarized by the two following results; note that in practice our criteria allow a complete determination of this image for two given elliptic conjugacy classes C_1 and C_2. Various examples of these image polygons were described and drawn in the last chapter of [Pau1] (pp. 137–141).

We will denote by $W_{red} \subset \mathbb{T}^2/\mathfrak{S}_2$ the image of all reducible groups (in the sense of linear representations); each line segment of W_{red} will be called a *wall*. The complete description of this reducible framework is given in [Pau2]. Now W_{red} **together with the two axes** $\{0\} \times S^1$ **and** $S^1 \times \{0\}$ disconnect $\mathbb{T}^2/\mathfrak{S}_2$ into a union of open convex polygons which we will call *chambers*. We will also call *totally reducible vertices* the two points which are the image of pairs (A, B) generating an Abelian group (i.e. having a common basis of eigenvectors).

Theorem 2.3 *Let C_1 and C_2 be two elliptic conjugacy classes in $PU(2,1)$, at least one of which is not a class of complex reflections. Then the image of the map $\tilde{\mu}$ in $\mathbb{T}^2/\mathfrak{S}_2$ is a union of closed chambers, containing in a neighborhood of each totally reducible vertex the convex hull of the reducible walls containing that vertex.*

This is reminiscent of the convexity theorem of Atiyah–Guillemin-Sternberg (see [A], [GS1], [GS2]), knowing that in this case reducible groups are what come closest to fixed points under the action of $PU(2,1)$ by conjugation (they have the smallest orbits).

Concerning the hypothesis on C_1 and C_2, it is easily seen that two complex reflections always generate a reducible group because their mirrors (fixed $\mathbb{C}$-planes) intersect in $\mathbb{C}P^2$, so in that case the image is a non-convex union of segments (see figure 4.12 of [Pau1]).

Note that these questions were also motivated by the study of Mostow's lattices in the respect that each of his families $\Gamma(p,t)$ with fixed p is part of our setting, sitting as a segment inside of the momentum polygon. This distinguishes a larger geometric family to which these lattices belong, and among which we hope to find other examples.

We have sharpened the results from this last chapter in [Pau2], where we obtained criteria to determine exactly which chambers are in the image of $\tilde{\mu}$. In particular we were able to determine when this map is onto:

Theorem 2.4 *Let C_1 and C_2 be two elliptic conjugacy classes in $PU(2,1)$, corresponding to angle pairs $\{\theta_1,\theta_2\}$ and $\{\theta_3,\theta_4\}$, with $\theta_i \in [0,2\pi[$, $\theta_1 \geqslant \theta_2$ and $\theta_3 \geqslant \theta_4$. Let:*

$$\tilde{\mu} : (C_1 \times C_2) \cap \mu^{-1}(\{\mathit{elliptics}\}) \xrightarrow{\mu} G \xrightarrow{\pi} \mathbb{T}^2/\mathfrak{S}_2$$

be the associated momentum map. Then:

$$\tilde{\mu} \text{ is onto} \iff \begin{cases} \theta_1 - 2\theta_2 + \theta_3 - 2\theta_4 \geqslant 2\pi \\ 2\theta_1 - \theta_2 + 2\theta_3 - \theta_4 \geqslant 6\pi \end{cases}$$

In contrast, there are no such conditions in $GL(3,\mathbb{C})$: if C_1, C_2 and C_3 are three semisimple conjugacy classes of $GL(3,\mathbb{C})$ then there exist matrices $M_i \in C_i$ satisfying $M_1M_2M_3 = Id$, by results of Simpson (see [Ko]). In other words, $\tilde{\mu}$ is always onto. Note that the semisimple conjugacy classes in $PU(2,1)$ are those of elliptic and loxodromic motions. Concerning loxodromic motions, our analysis implies that if C_1 and C_2 are two elliptic conjugacy classes in $PU(2,1)$ (at least one of which is not a class of $\mathbb{C}$-reflections), then the product AB can take values in any loxodromic conjugacy class as (A,B) varies in $C_1 \times C_2$ (Falbel and Wentworth obtain the analogous result in [FW2] when C_1 and C_2 are themselves loxodromic conjugacy classes). This follows from the fact that the image of reducible pairs does not disconnect the space of loxodromic conjugacy classes of $PU(2,1)$ (thus there is only one chamber).

3 Current projects

My main goal is to obtain new discrete subgroups of $PU(2,1)$. The problems which arise are the following. First, we need a method to produce good candidates: such a method is obtained from the study of angle pair configurations in the last part of my thesis. Then, given a group with explicit matrix generators, it remains to determine whether it is (contained in) an arithmetic lattice. If it isn't, then discreteness is a delicate question. There are methods to convince oneself that the group indeed has good chances of being discrete, such as systematically investigating words of a given maximal length (cf. Schwartz), or using Deraux's algorithm based on the Dirichlet method. If these first tests are convincing, one tries to build a fundamental domain, using among others techniques developed in the second part of my thesis ([DFP]). This will give us some more information on the group, such as a presentation and the volume of the quotient orbifold (if it is finite). It then remains to determine whether or not the group is really new, i.e. not commensurable to any lattice in the list of Deligne-Mostow or Thurston.

In concrete terms, I am presently looking for new groups in configurations of the Mostow type, that is groups generated by a complex reflection of order p and a cyclic permutation of order 3 (denoted by J above).

3.1 Discrete groups generated by higher order complex reflections

This part is joint work with John Parker ([ParPau]). We consider symmetric complex hyperbolic triangle groups generated by three complex reflections with angle $2\pi/p$. We restrict our attention to those groups where certain words are elliptic. Our goal is to find necessary conditions for such a group to be discrete. The main application we have in mind is that such groups are candidates for non-arithmetic lattices.

In [Mos1] Mostow constructed the first examples of non-arithmetic complex hyperbolic lattices. These lattices were generated by three complex reflections R_1, R_2 and R_3 with the property that there exists a complex hyperbolic isometry J of order 3 so that $R_{j+1} = JR_jR^{-1}$. In Mostow's examples the generators R_j have order $p = 3$, 4 or 5. Subsequently Deligne and Mostow constructed further non-arithmetic lattices as monodromy groups of certain hypergeometric functions [DM] (these lattices were known to Picard who did not consider their arithmetic nature). These lattices are (commensurable with) groups generated by complex reflections R_j with other values of p; see Mostow [Mos2] and Sauter [Sa]. Subsequently no new non-arithmetic lattices have been constructed.

In [Par2] Parker considered the case $p = 2$. That is, he considered complex involutions I_1, I_2 and I_3 with the property that there is a J of order 3 so that $I_{j+1} = JI_jJ^{-1}$. In particular, he used a theorem of Conway and Jones [CJ] to classify all such groups where I_1I_2 and $I_1I_2I_3$ are elliptic.

Remarkably, when $p \geq 3$ finding groups for which R_1R_2 and $R_1R_2R_3$ are elliptic involves solving the same equation as in the case $p = 2$. In this paper we use the solutions to this equation found using [CJ] in [Par2] in the general case.

We describe the configuration space of all groups generated by a complex reflection R_1 of order p and a regular elliptic motion J of order 3. This configuration space is parametrized by the conjugacy class of the product R_1J, which we represent geometrically in two different manners. The first, following Goldman, Parker, is to consider the trace of R_1J; this determines the conjugacy class of R_1J when it is loxodromic, but there is a threefold indetermination when it is elliptic or parabolic. The second manner, following Paupert, is to use the geometric invariants of the conjugacy class, i.e. an angle pair for elliptic isometries and a pair (angle, length) for loxodromic isometries. We will use both parameter spaces in this paper, where we focus on the elliptic case.

Our first result is the direct analogue of the main theorem of [Par2], and can be roughly stated as follows:

Theorem 3.1 *Let R_1 be a complex reflection of order p and J a regular elliptic of order 3 in $PU(2,1)$. Suppose that R_1J and $R_1R_2 = R_1JR_1J^{-1}$ are elliptic. If the group $\Gamma = \langle R_1, R_2, R_3\rangle$ is discrete then one of the following is true:*

- Γ *is one of Mostow's lattices.*
- Γ *is a normal subgroup of one of Mostow's groups.*
- Γ *is one of the sporadic groups described below.*

The sporadic groups correspond to the 18 exceptional solutions from [CJ], which do not depend on p (the groups do change with p of course). We determine for each $p \geq 3$ which of these points lie inside our configuration space. One must then analyze each of these groups to decide whether or not it is discrete, if so whether or not it is a lattice and if so whether or not it is arithmetic. We illustrate ways to attack this problem by showing that certain solutions are arithmetic and certain other solutions are non-discrete. We analyze in detail the situation for $p = 3$, which can be summarized as follows:

Theorem 3.2 *There are 16 sporadic groups for $p = 3$, with the following properties:*

- *Four of them fix a point in $H^2_{\mathbb{C}}$.*
- *One stabilises a complex line.*
- *One is contained in an arithmetic lattice.*
- *The ten remaining groups are none of the above.*

The crucial question is then to determine whether or not the ten remaining groups are discrete. We give a negative answer for three of them, by finding elliptic elements of infinite order in the group.

3.2 Fundamental domains for the sporadic complex reflection groups

It remains to complete the project outlined above. The next step is to find out which of these sporadic groups are discrete, and if yes which of them are lattices. Note that we have an infinite family of groups to study (some of the sporadic solutions are in the configuration space for all values of p).

The most reasonable way to approach this problem is to construct fundamental domains for their action in $H^2_{\mathbb{C}}$, in the setting of the Poincaré polyhedron theorem. This requires some work (at least a few months), but will answer both questions at once. It will also give us information about the groups contained in arithmetic lattices (do they have finite index? if yes, what is the index?). In fact we can use essentially the same construction as in [DFP], the two generators being of the same form. We have already analyzed the conjugacy class of crucial words in the group (such as the products R_1R_2, the braiding words $R_1R_2R_1R_2^{-1}R_1^{-1}R_2^{-1}$, and the "Mostow words" R_2R_1J and $J^{-1}R_1R_2$). In three cases we obtain elliptic words of infinite order (and thus non-discreteness of the group), but in the remaining cases the preliminary discreteness tests are positive.

4 Projects for a near future

4.1 Reflection groups

Apart from purely loxodromic groups (arising for instance by deformation of Fuchsian groups), the only known discrete groups in $Isom(H^n_{\mathbb{C}})$ are of this type. Recall that there are two types of reflections in $Isom(H^n_{\mathbb{C}})$, $\mathbb{C}$-reflections which are holomorphic isometries fixing pointwise a totally geodesic complex hypersurface (a copy of $H^{n-1}_{\mathbb{C}} \subset H^n_{\mathbb{C}}$), and $\mathbb{R}$-reflections which are antiholomorphic isometries fixing pointwise a Lagrangian subspace (a copy of $H^n_{\mathbb{R}} \subset H^n_{\mathbb{C}}$). There are many open problems concerning reflection groups in $Isom(H^n_{\mathbb{C}})$, among which we can address the following:

- **Finite reflection groups:**
  Finite $\mathbb{C}$-reflection groups have been classified by Shephard and Todd (see also Broué–Malle–Rouquier). In [FPau] we have proven that all finite subgroups of $U(2)$ are (of index 2 in a group) generated by $\mathbb{R}$-reflections. Is this true in $U(n)$? Probably not. The question is then to classify such subgroups of $U(n)$.

- **Lattices generated by reflections in higher dimensions:**

  In real hyperbolic space, it is known that lattices generated by reflections only exist in small dimensions. Vinberg has proven that there are no compact Coxeter polyhedra in $H^n_{\mathbb{R}}$ for $n \geqslant 30$, and Prokhorov has proven that there are no Coxeter polyhedra of finite volume in $H^n_{\mathbb{R}}$ for $n \geqslant 996$ (the known examples are for $n \leqslant 8$ in ther first case and $n \leqslant 21$ in the second).

  Is the situation analogous in $PU(n,1)$? The known examples (Deligne–Mostow, Mostow, Allcock) are in dimension $n \leqslant 13$ (in fact, $n \leqslant 9$ except for one of Allcock's examples). An obvious difficulty is that there is no counterpart of Coxeter polyhedra.

- **Non-arithmetic lattices generated by reflections:**

  Non-arithmetic lattices in $PU(n,1)$ are only known for $n = 2$ (14 lattices due to Picard, Mostow, Deligne–Mostow) and $n = 3$ (one non-cocompact example due to Deligne–Mostow). We hope to obtain new non-arithmetic lattices in $PU(2,1)$ in the families of symmetric $\mathbb{C}$-reflection triangle groups described above. We can hope to apply our methods in dimension 3 and maybe 4, but after that the direct geometric method (construction of fundamental domains) becomes hopeless.

4.2 Discrete groups generated by regular elliptic motions

We can also explore the general case of groups generated by two regular elliptic motions (ie not $\mathbb{C}$-reflections), where the parameter space is much bigger. The problem of finding discrete groups in this parameter space seems hopeless, but the methods from the last part of my thesis allow to determine some "good" one-parameter families to investigate. Precisely, if the three angle pairs of the elliptic motions A, B, and AB are prescribed, then there is a one-parameter family of such groups generated by Lagrangian (or real) reflections, and this allows to search systematically such families.

5 Related questions

- Representation spaces of surface groups in $PU(2,1)$ and complex hyperbolic quasi-Fuchsian groups. Character varieties of 3-manifolds in $PU(2,1)$.
- Existence of spherical CR structures on 3-manifolds (Schwartz, Falbel)
- Fake projective planes (Mumford, Klingler, Yeung, Prasad)
- Complex hyperbolic structures on moduli spaces of algebraic objects (Allcock–Carlson–Toledo)
- Negatively curved compact Kähler manifolds not covered by the ball (Mostow–Siu, Deraux)
- Spectrum of the automorphic Laplace-Beltrami operator on arithmetic ball quotients (Francsics–Lax, Sarnak)

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