Problem 5. If \( f \) is differentiable on \((a, b)\) and \( f'(x) \neq 0 \) for all \( x \) in the interval, prove that either \( f'(x) > 0 \) or \( f'(x) < 0 \) on the entire interval.

Solution. This is a direct corollary of the intermediate value property of the derivative of a differentiable function. If \( f'(x_1) < 0 \) and \( f'(x_2) > 0 \) for some \( x_1, x_2 \in (a, b) \) then there must be a point \( x_3 \in (x_1, x_2) \) such that \( f'(x_3) = 0 \), a contradiction. \( \square \)

Problem 11. Show that the class of rational functions (polynomial divided by polynomial) is closed under the operation of differentiation.

Solution. This is a simple application of the quotient formula
\[
\left( \frac{f}{g} \right)' = \frac{fg' - f'g}{g^2}
\]
and the fact that the derivative of a polynomial, the product of two polynomials are also polynomials. \( \square \)

Problem 13. Let \( f_n \) denote the \( n \)th iterate of \( f, f_1 = f, f_2(x) = f(f(x)), \ldots, f_n(x) = f(f_{n-1}(x)). \) Express \( f_n' \) in terms of \( f' \). Show that if \( a \leq |f(x)| \leq b \) for all \( x \), then \( a^n \leq |f(x)| \leq b^n. \)

Solution. This is an iterative application of the chain rule of differentiation. In other words we will have
\[
\frac{d}{dx} f_n(x) = f'(f_{n-1}(x)) \cdot f'(f_{n-2}(x)) \cdots f'(x).
\]
Now the second part will follow immediately. \( \square \)

Problem 1. Suppose \( f \) is a \( C^2 \) function on an interval \((a, b)\) and the graph of \( f \) lies above every secant line. Prove that \( f'(x) \leq 0 \) on the interval.

Solution. From the assumption we will have the following crucial inequality:
\[
\frac{f(x_1) - f(x_2)}{x_1 - x_2} \geq \frac{f(x_2) - f(x_3)}{x_2 - x_3} \geq \frac{f(x_3) - f(x_4)}{x_3 - x_4}, \ a < x_1 < x_2 < x_3 < x_4 < b.
\]
Now we let \( x_1 \to x_2 \) and \( x_4 \to x_3 \), we will have, if \( x_2 < x_3 \), then \( f'(x_2) \geq f'(x_3) \), which means the derivative of \( f \) is a decreasing function, therefore \( f''(x) \leq 0. \) \( \square \)

Problem 3. If \( f \) is a \( C^2 \) function on an interval, prove that
\[
\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x).
\]

Solution. This is a typical question which shows that sometimes the mean value theorem is not sufficient to prove certain desired properties of the functions we are investigating. For
this problem, you can surely use mean value theorem at the first step, but this is not taking us much close to the final goal, because we don’t know where the “mean value” point is exactly located.

Therefore we need the Taylor formula which gives finer information of $f$ up to the second order:

$$f(x + h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + o(h^2).$$

By considering this expansion and the counterpart for $f(x - h)$ we immediately proved this problem. \qed

**Problem 14.** If $f(x) = 1/(1 + x)$ show that $T_n(f, 0, x) = \sum_{k=0}^{n}(-1)^kx^k$.

**Solution.** Apparently we need only to compute all the derivatives $f^{(k)}(0)$. In fact because we have

$$f(x) = (1 + x)^{-1}$$

a simple calculations shows

$$f^k(x) = (-1)(-2)\cdots(-k)(1 + x)^{-k-1} = (-1)^k k! (1 + x)^{-k-1}.$$ \qed

**Problem 23.**

**Solution.**

(a). Use mean value theorem $n$ times for the functions $f, f', f'', \ldots$. More specifically, if we have

$$f(x_1) = f(x_2) = \cdots = f(x_{n+1}) = 0$$

then we have $x_{1k} \in [x_k, x_{k+1}], k = 1, 2, \ldots, n$ such that

$$f'(x_{11}) = f'(x_{12}) = \cdots = f'(x_{1n}) = 0.$$ Repeating this process $n$ times completes the proof.

(b). This is just the reversed statement of the previous part.

(c). A direct corollary of part (b).

(d). Apply part (a) to the following function

$$f(x)(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) - f(x_1)(x - x_2)(x - x_3)(x_2 - x_3)
- f(x_2)(x_1 - x)(x - x_3)(x_1 - x_3) - f(x_3)(x_1 - x)(x_2 - x)(x_1 - x_2)$$

\qed