Problem 1. Consider the integrable 2π-periodic function

\[ f(\theta) = \frac{\pi - \theta}{2} \quad \theta \in [0, 2\pi). \]

1. Compute the Fourier series for \( f(\theta) \).
2. Discuss the convergence of the above Fourier series. More precisely, determine at which point(s) \( \theta \) the Fourier series converges to \( f(\theta) \), at which point(s) it does not, and over which interval(s) this convergence is uniform. Justify your answers.

Solution. By definition, we have

\[ \hat{f}(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) e^{-in\theta} d\theta = \frac{1}{4} \int_{0}^{2\pi} e^{-in\theta} d\theta - \frac{1}{4\pi} \int_{0}^{2\pi} \theta e^{-in\theta} d\theta. \]

If \( n = 0 \), then we have

\[ \hat{f}(0) = \frac{1}{4} \int_{0}^{2\pi} d\theta - \frac{1}{4\pi} \int_{0}^{2\pi} \theta d\theta = \frac{\pi}{2} - \frac{1}{4\pi} \frac{(2\pi)^2 - 0^2}{2} = 0. \]

If \( n \neq 0 \), then we have

\[ \hat{f}(n) = -\frac{1}{4\pi} \int_{0}^{2\pi} \theta e^{-in\theta} d\theta = \frac{1}{4\pi} \frac{1}{in} \left[ \theta e^{-in\theta} \right]_{0}^{2\pi} - \frac{1}{4\pi} \int_{0}^{2\pi} e^{-in\theta} d\theta = \frac{1}{4\pi} \frac{1}{in} \cdot (2\pi) = \frac{1}{2in}. \]

Hence we have the Fourier series

\[ f(\theta) \sim \sum_{n=-\infty \atop n \neq 0}^{\infty} \hat{f}(n) e^{in\theta} = \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{e^{in\theta}}{2in} = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}. \]

We note \( f(\theta) \) is not continuous at the points \( 2\pi\mathbb{Z} \) and is of class \( C^2 \) at any other points. Further, the Fourier series evaluated at \( \theta \) obviously is equal to 0, while \( f(\theta) = \frac{\pi}{2} \). Hence the Fourier series for \( f(\theta) \) converges to \( f(\theta) \) for every \( \theta \notin 2\pi\mathbb{Z} \), and this convergence is uniform over every closed interval disjoint with \( 2\pi\mathbb{Z} \). □

Remark 0.1. A common mistake in Problem 1(a) is to overlook the different treatments for \( n = 0 \) and \( n \neq 0 \). This can happen to even experts, and can lead to disastrous results, so be careful!
Problem 2. Keep considering the integrable $2\pi$-periodic function

$$f(\theta) = \frac{\pi - \theta}{2} \quad \theta \in [0, 2\pi).$$

Also, consider the integrable $2\pi$-periodic function

$$g(\theta) = \frac{\pi - \theta}{2} \quad \theta \in [-\pi, \pi).$$

(1) Compute the convolution $h(\theta) = (f * g)(\theta)$.

(2) Compute the Fourier series for $h(\theta)$.

**Solution.** By periodicity, it suffices for us to compute $h(\theta)$ for $\theta \in [0, 2\pi)$. Since

$$g(\theta) = g(\theta + 2\pi) = \frac{\pi - (\theta + 2\pi)}{2} = -\frac{\pi + \theta}{2} \quad \theta \in [-3\pi, -\pi),$$

$$g(\theta) = g(\theta - 2\pi) = \frac{\pi - (\theta - 2\pi)}{2} = \frac{3\pi - \theta}{2} \quad \theta \in [\pi, 3\pi),$$

for $0 \leq \theta < \pi$ we have

$$h(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(\theta - x) \, dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi - x}{2} g(\theta - x) \, dx$$

$$= \frac{1}{2\pi} \int_0^{\theta + \pi} \frac{\pi - x}{2} g(\theta - x) \, dx + \frac{1}{2\pi} \int_{\theta + \pi}^{2\pi} \frac{\pi - x}{2} g(\theta - x) \, dx$$

$$= \frac{1}{2\pi} \int_0^{\theta + \pi} \frac{\pi - x}{2} \pi - x - \pi + x \, dx - \frac{1}{2\pi} \int_{\theta + \pi}^{2\pi} \frac{\pi - x}{2} \pi - x + \pi - x \, dx$$

$$= \frac{\pi^2 - 3\theta^2}{24},$$

and for $\pi \leq \theta < 2\pi$ we have

$$h(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(\theta - x) \, dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{\pi - x}{2} g(\theta - x) \, dx$$

$$= \frac{1}{2\pi} \int_0^{\theta - \pi} \frac{\pi - x}{2} g(\theta - x) \, dx + \frac{1}{2\pi} \int_{\theta - \pi}^{2\pi} \frac{\pi - x}{2} g(\theta - x) \, dx$$

$$= \frac{1}{2\pi} \int_0^{\theta - \pi} \frac{\pi - x}{2} \pi - x - 3\pi + \theta + x \, dx + \frac{1}{2\pi} \int_{\theta - \pi}^{2\pi} \frac{\pi - x}{2} \pi - x + \pi - \theta + x \, dx$$

$$= -\frac{3\theta^2 - 12\pi\theta + 11\pi^2}{24}.$$

Hence

$$\hat{h}(n) = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) e^{-in\theta} \, d\theta = \frac{1}{2\pi} \int_0^{\pi} h(\theta) e^{-in\theta} \, d\theta + \frac{1}{2\pi} \int_{\pi}^{2\pi} h(\theta) e^{-in\theta} \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{\pi} \frac{\pi^2 - 3\theta^2}{24} e^{-in\theta} \, d\theta - \frac{1}{2\pi} \int_{\pi}^{2\pi} \frac{3\theta^2 - 12\pi\theta + 11\pi^2}{24} e^{-in\theta} \, d\theta$$

$$= \begin{cases} 0, & \text{if } n = 0; \\ -\frac{(-1)^n}{4\pi}, & \text{if } n \neq 0, \end{cases}$$
and we have
\[ h(\theta) \sim - \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{4n^2} e^{in\theta} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta). \]

**Remark 0.2.** Alternatively, we can compute \( \hat{h}(n) \) without determining \( h(\theta) \) first. In fact, since \( h = f * g \), we have \( \hat{h}(n) = \hat{f}(n)\hat{g}(n) \). Here \( \hat{f}(n) \) has been obtained in Problem 1(a), and we have (this can be also obtained by observing \( g(\theta) = f(\theta + \pi) + \frac{\pi}{2} \); how?)

\[
\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi - \theta}{2} e^{-in\theta} d\theta = \begin{cases} \frac{\pi}{2}, & \text{if } n = 0; \\ \frac{(-1)^n}{2n^1}, & \text{if } n \neq 0. \\ \end{cases}
\]

Hence
\[
\hat{h}(n) = \hat{f}(n)\hat{g}(n) = \begin{cases} 0, & \text{if } n = 0; \\ \frac{(-1)^n}{4n^4}, & \text{if } n \neq 0. \\ \end{cases}
\]

You can stop here, or you may observe that
\[
\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty,
\]
so the Fourier series
\[
\sum_{n=-\infty}^{\infty} \hat{h}(n)e^{in\theta} = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{4n^2} e^{in\theta} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta)
\]
defines a continuous function and shares the same Fourier coefficient with \( h \). Since \( h \), as a convolution, is also continuous everywhere, they must be equal to each other, and this gives
\[
h(\theta) = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{4n^2} e^{in\theta} = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\theta).
\]

**Problem 3.** Insist on considering the integrable \( 2\pi \)-periodic function

\[ f(\theta) = \frac{\pi - \theta}{2}, \quad \theta \in [0, 2\pi). \]

1. State the requirements for good kernels, and verify whether the family of functions \( \{f(\theta), f(2\theta), f(3\theta), \ldots\} \) is a family of good kernels.
2. Compute the Abel sum for the infinite series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \).

**Solution.** A family of integrable \( 2\pi \)-periodic functions \( \{K_n\} \) is called a family of good kernels if

1. \( \frac{1}{2\pi} \int_{0}^{2\pi} K_n(x) \, dx = 1; \)
2. \( \frac{1}{2\pi} \int_{0}^{2\pi} |K_n(x)| \, dx \) is uniformly bounded for \( n \geq 1 \);
3. for every \( \delta > 0 \) we have \( \lim_{n \to \infty} \frac{1}{2\pi} \int_{\delta < |x| < \pi} |K_n(x)| \, dx = 0. \)

\[ \text{[Yes, this is a fancy way for saying that there exists an absolute constant such that]} \]
\[ \frac{1}{2\pi} \int_{0}^{2\pi} |K_n(x)| \, dx < M. \]
The family \( \{ f(\theta), f(2\theta), \ldots \} \) is not a family of good kernels, as
\[
\frac{1}{2\pi} \int_{0}^{2\pi} f(x) \, dx = \frac{1}{2\pi} \int_{0}^{\frac{2\pi}{2}} \frac{\pi - x}{2} \, dx = 0 \neq 1.
\]

As for the Abel sum, write
\[
A(r) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} r^n.
\]

Obviously \( A(r) \) is absolutely convergent for \( 0 \leq r < 1 \), and
\[
A'(r) = \sum_{n=1}^{\infty} (-1)^n r^{n-1} = -\sum_{n=0}^{\infty} (-r)^n = -\frac{1}{1+r},
\]
\[
A(r) = \int_{0}^{r} A'(t) \, dt + A(0) = -\int_{0}^{r} \frac{dt}{1+t} = -\ln(1+r).
\]

Hence the Abel sum for \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) is
\[
\lim_{r \to 1^0} A_r = -\lim_{r \to 1^0} \ln(1+r) = -\ln 2. \quad \square
\]

**Remark 0.3.** How about the Abel sum for \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \)? This is what was originally in my mind.

**Problem 4.** Use Fourier series to solve the following wave equation
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \sin(2\pi x), \quad \frac{\partial u(x,0)}{\partial t} = \sin(4\pi x) \quad (0 \leq x \leq 1).
\]

**Solution.** We recall that our classical discussion for the wave equation is always for the period \( \pi \), so we have to make a change of variables \( X = \pi x, T = \pi t \) to get
\[
\frac{\partial^2 U}{\partial T^2} = \frac{\partial^2 U}{\partial X^2}, \quad U(X,0) = \sin(2X), \quad \frac{\partial U(X,0)}{\partial T} = \frac{1}{\pi} \sin(4X) \quad (0 \leq X \leq \pi).
\]

Hence
\[
U(X,T) = \sum_{n=-\infty}^{\infty} \left( c_n \cos(nT) + \frac{d_n}{n} \sin(nT) \right) \sin(nX),
\]
where \( c_n \) and \( d_n \) are the Fourier coefficients (in terms of the sine function) for \( \sin(2X) \) and \( \sin(4X) \) respectively, so
\[
c_n = \begin{cases} 1, & \text{if } n = 2; \\ 0, & \text{if } n \neq 2; \end{cases} \quad d_n = \begin{cases} \frac{1}{\pi}, & \text{if } n = 4; \\ 0, & \text{if } n \neq 4. \end{cases}
\]

Hence
\[
U(X,T) = \cos(2T) \sin(2X) + \frac{\sin(4T) \sin(4X)}{4\pi},
\]
\[
u(x,t) = \cos(2\pi t) \sin(2\pi x) + \frac{\sin(4\pi t) \sin(4\pi x)}{4\pi}. \quad \square
Remark 0.4. The answer may be double checked applying d’Alembert’s formula
\[
    u(x, t) = \frac{\sin(2\pi(x + t)) + \sin(2\pi(x - t))}{2} + \frac{1}{2} \int_{x-t}^{x+t} \sin(4\pi y) \, dy \\
    = \cos(2\pi t) \sin(2\pi x) - \frac{\cos(4\pi(x + t)) - \cos(4\pi(x - t))}{8\pi} \\
    = \cos(2\pi t) \sin(2\pi x) + \frac{\sin(4\pi t) \sin(4\pi x)}{4\pi}.
\]
However, since the problem has specified to “use Fourier series to solve the following wave equation”, we cannot use this approach as the official solution to the problem.

**Problem 5.** Let \( f \) be an integrable \( 2\pi \)-periodic function. Show that
\[
    \| f \|_2^2 = \| f - S_N(f) \|_2^2 + \| S_N(f) \|_2^2.
\]

**Proof.** We have
\[
    \| f \|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_0^{2\pi} |(f(x) - S_N(f)(x)) + S_N(f)(x)|^2 \, dx \\
    = \frac{1}{2\pi} \int_0^{2\pi} |f(x) - S_N(f)(f)(x)|^2 \, dx + \frac{1}{2\pi} \int_0^{2\pi} |S_N(f)(f)(x)|^2 \, dx \\
    + \frac{2}{2\pi} \int_0^{2\pi} (f(x) - S_N(f)(f)(x))S_N(f)(f)(x) \, dx \\
    = \| f - S_N(f) \|_2^2 + \| S_N(f) \|_2^2 + \frac{1}{\pi} \sum_{n=-N}^{N} \hat{f}(n) \left( \int_0^{2\pi} f(x)e^{inx} \, dx - \int_0^{2\pi} S_N(f)(f)(x)e^{inx} \, dx \right) \\
    = \| f - S_N(f) \|_2^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2. \]

Remark 0.5. Some students applied the abstract theory of inner products and orthonormal bases. This approach is also fine, but then they showed some insufficient understanding in their arguments, just copying certain sentences and terminologies from memory, and this is not fine. We should always know very clearly what we are talking about!