LECTURE 5. QUADRATIC MAPS

DYNAMICAL SYSTEMS (110.421)
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In this class, we consider the discrete dynamical system

$$x_{n+1} = f(x_n) \tag{*}$$

given by the quadratic map

$$f : [0,1] \to [0,1] \qquad x \mapsto \lambda x(1-x), \tag{0.1}$$

where $0 \le \lambda \le 4$ is the parameter.

1. The Fixed Points

Theorem 1.1. *Consider the dynamical system* (*).

- *If* $0 \le \lambda \le 1$*, then the system has a unique fixed point* 0*.*
- *If* $\lambda > 1$*, then the system has exactly two fixed points* 0 *and* $x_\lambda = 1 - \frac{1}{\lambda}$*.*

2. The Asymptotic Orbit Behaviors

Theorem 2.1. *Consider the dynamical system* (*).

- *If* $0 \le \lambda \le 1$*, then all the orbits over* $[0,1]$ *are asymptotic to* 0*.*
- *If* $1 < \lambda \le 3$*, then all the orbits over* $(0,1)$ *are asymptotic to* $x_\lambda = 1 - \frac{1}{\lambda}$*, while the orbits for* 0 *and* 1 *are (trivially) asymptotic to* 0*.*

Remark 2.2. To show the convergence of an orbit, usually we have three approaches.

1. Show that the map $f(x)$ is contracting, usually by showing $|f'(x)| \le \lambda < 1$, so that all the orbits converge to the unique fixed point.
2. Show that the map $f(x)$ is piecewise monotone, usually by showing $f'(x) > 0$ (or $f'(x) < 0$), so that every orbit converges to a fixed point (though different orbits may converge to different fixed points).
3. Show that the orbit is a bounded increasing/decreasing sequence, usually by showing $f(x) > x$ (or $f(x) < x$). This is in fact a special case of the above approach, but can be very convenient in many situations.

To apply these approaches, quite often we need to verify that $f(x)$ (or $f'(x)$) satisfies certain criteria in terms of inequalities, and this requires in turn the computation of the maximum/minimum of $f(x)$ (or $f'(x)$) over intervals. If you do not remember clearly how to compute extremas of real functions, check your calculus textbook.

Remark 2.3. Once λ is greater than 3, besides the fixed points 0 and x_λ we also have a period-2 orbit for the dynamical system (*). In fact, given an initial state $x \in [0,1]$, unless $x = 0, x_\lambda, 1$, the orbit of x will inevitably converges to this period-2 orbit. This phenomenon is quite common in real life. For example, the populations of many insects, the yields of plants, or the traffics on highways, often demonstrate such alternative up-and-down cycles. When λ gets even larger, then very quickly a period-4 attracting orbit is thrown into the picture, and then a period-8 attracting orbit, and then a period-16 attracting orbit, ..., until the system eventually becomes chaotic.

3. Homework

In this class, we have learned to

- apply various approach to study the fixed points and asymptotic orbit behaviors;
- understand the different behaviors of quadratic dynamical systems with respect to different parameters.

Today's homework is 2.5.1, 2.5.2 and the following exercise.

Exercise 3.1. Consider the dynamical system $x_{n+1} = f(x_n)$ given by the quadratic map

$$f : [0, 1] \to [0, 1], \qquad x \mapsto 1 - \mu x^2, \tag{3.1}$$

where $\mu \geq 0$ is a parameter. In this case, what should Theorem 2.1 look like? Clearly state your version of the theorem and prove your claim by mimicking that of Theorem 2.1.

Hint. Exercise 2.5.2 may (or may not?) be helpful. □