

## LECTURE 8. LINEAR MAP. II

DYNAMICAL SYSTEMS (110.421)  
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### 1. DYNAMICAL BEHAVIORS OF LINEAR MAPS

**Theorem 1.1.** Consider a 2-dimensional discrete linear dynamical system

$$(1.1) \quad x_{n+1} = Ax_n,$$

where  $A$  is a  $2 \times 2$  real matrix, and let  $\lambda_1, \lambda_2$  be the eigenvalues of  $A$ .

- (1) If  $|\lambda_1| < 1$  and  $|\lambda_2| < 1$ , then all the orbits converges to 0.
- (2) If  $|\lambda_1| > 1$  or  $|\lambda_2| > 1$ , then, with obvious exceptions, all the orbits diverges to  $\infty$ .

*Remark 1.2.* As we discussed in class, in the above cases we can also easily determine the trajectories of the orbits.

*Remark 1.3.* The situation for higher dimensional linear dynamical systems are essentially the same, as long as we apply the theory of Jordan normal forms. It is a good exercise (not homework!) to write out a complete classification theory for three-dimensional linear dynamical systems.

### 2. APPLICATIONS: LINEARLY RECURSIVE SEQUENCES

**Theorem 2.1.** Consider a linearly recursive sequence  $\{a_0, a_1, \dots\}$  given by

$$(2.1) \quad a_{n+1} = \alpha_1 a_n + \dots + \alpha_r a_{n+1-r},$$

where  $r \geq 1$  and  $\alpha_1, \dots, \alpha_r$  are constants. Let  $\lambda_1, \dots, \lambda_r$  be the roots to the equation

$$(2.2) \quad x^r = \alpha_1 x^{r-1} + \dots + \alpha_{r-1} x + \alpha_r,$$

and assume that they are distinct. Then we have the general formula

$$(2.3) \quad a_n = \beta_1 \lambda_1^n + \dots + \beta_r \lambda_r^n,$$

where  $\beta_1, \dots, \beta_r$  are suitable constants.

*Sketch of Proof.* Let

$$x_n = \begin{pmatrix} a_{n+r} \\ a_{n+r-1} \\ \dots \\ a_{n+1-r} \end{pmatrix}$$

and consider the  $r$ -dimensional discrete linear dynamical system

$$x_n = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{r-1} & \alpha_r \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} x_{n-1} = Ax_{n-1},$$

say. We have

$$\det(xI_r - A) = \begin{vmatrix} x - \alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_{r-1} & -\alpha_r \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & x \end{vmatrix} = x^r - \alpha_1 x^{r-1} - \alpha_2 x^{r-2} - \cdots - \alpha_r,$$

so the eigenvalues for  $A$  are exactly  $\lambda_1, \dots, \lambda_r$ . By assumption they are distinct, so  $A$  is similar to a diagonal matrix, namely there exists an  $r \times r$  invertible matrix  $S$

such that  $A = S^{-1} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} S$ . This implies that

$$x_n = Ax_{n-1} = A^n x_0 = S^{-1} \begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_r^n \end{pmatrix} S x_0.$$

Hence  $a_{n+r}$ , as the first entry of  $x_n$ , is a linear combination of  $\lambda_1^n, \dots, \lambda_r^n$ , and so of  $\lambda_1^{n+r}, \dots, \lambda_r^{n+r}$ .  $\square$

### 3. HOMEWORK

In this class, we have learned to

- explicitly determine the dynamical behavior for 2-dimensional linear dynamical systems;
- determine, in principle, the dynamical behavior for high-dimensional linear dynamical systems;
- explicitly determine the general formula for linearly recursive sequences.

Today's homework is 3.1.5 and the following exercises.

**Exercise 3.1.** In the second part of Theorem 1.1, there are some “obvious exceptions”. Explain what these exceptions are (in terms of various quantities associated to  $A$ ).

**Exercise 3.2.** Complement Theorem 1.1 with a discussion of the cases when, say,  $|\lambda_1| = 1$  and  $|\lambda_2| \leq 1$ . More precisely, in each case, determine the convergence/divergence of various orbits, as well as all the fixed and/or periodic points.