LECTURE 12. INVERTIBLE CIRCLE MAPS

DYNAMICAL SYSTEMS (110.421)
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In this lecture, for every $x \in \mathbb{R}$, we denote by $[x]$ the corresponding point on S^1 subject to the identification $S^1 = \mathbb{R}/\mathbb{Z}$.

1. Lifts of Circle Maps

Let $f : S^1 \to S^1$ be a circle map, then a map $F : \mathbb{R} \to \mathbb{R}$ is called a *lift* of f if lift

$$f([x]) = [F(x)] \qquad (x \in \mathbb{R}). \tag{1.1}$$

Theorem 1.1. *Let $f : S^1 \to S^1$ be a continuous circle map, then*

1. *there always exists a continuous lift of f;*
2. *the continuous lift of f is unique up to integers.*

Theorem 1.2. *Let $f : S^1 \to S^1$ be a continuous circle map, $F : \mathbb{R} \to \mathbb{R}$ a continuous lift of f, and write*

$$\deg(f) = F(x+1) - F(x) \tag{1.2}$$

Then

1. $\deg(f)$ *is an integer;*
2. $\deg(f)$ *is independent of the choices of both x and F;*
3. *f is homeomorphism only if* $\deg(f) = \pm 1$.

Let $f : S^1 \to S^1$ be a continuous circle map, then

$$\deg(f) = F(x+1) - F(x) \tag{1.3}$$

is called the *degree* of f, where $F : \mathbb{R} \to \mathbb{R}$ any continuous lift of f and $x \in \mathbb{R}$. In particular, f is called *orientation-preserving* if $\deg f = 1$, or *orientation-reversing* if $\deg(f) = -1$. degree orientation-preserving orientation-reversing

2. Rotation Numbers

Theorem 2.1. *Let $f : S^1 \to S^1$ be a continuous circle map, $F : \mathbb{R} \to \mathbb{R}$ a continuous lift of f, and write*

$$\rho(F) = \lim_{n \to \pm\infty} \frac{F^n(x) - x}{n}. \tag{2.1}$$

Then

1. *the limit always exists, and its value is independent of the choice of x;*
2. *$\rho(F) \in \mathbb{Q}$ if and only if f has a periodic point.*

Let $f : S^1 \to S^1$ be a continuous circle map, then

$$\rho(f) = [\rho(F)] \tag{2.2}$$

is called the *rotation number* of f, where $F : \mathbb{R} \to \mathbb{R}$ is any continuous lift of f. rotation number

3. Dynamical Behaviors

Roughly speaking, the dynamical behavior of f is asymptotically similar to that of $R_{\rho(f)}$. An important difference in our case is the loss of homogeneity, so not all the orbits share the same kind of dynamical behaviors. Instead, some special orbits stand out and become very "attractive".

Theorem 3.1. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism. Assume that $\rho(f) \in \mathbb{Q}$, say $\rho(f) = [\frac{p}{q}]$ for some relatively prime integers p and q with $q \geq 1$. Then*

1. *the periodic orbits of f all have the prime period q;*
2. *for every periodic point $[x] \in S^1$ we have $F^q(x) = x + p$;*
3. *the ordering of $\{x, f(x), \dots, f^{q-1}(x)\}$ on S^1 is the same as that of $\{[0], [\frac{p}{q}], [\frac{2p}{q}], \dots, \frac{(q-1)p}{q}]\}$;*
4. *under the action of f^q, every nonperiodic orbit is heteroclinic to two points on a periodic orbit.*

Remark 3.2. In Part 3 of Theorem 3.1, we quoted the ordering of the points $\{[0], [\frac{p}{q}], [\frac{2p}{q}], \dots, \frac{(q-1)p}{q}]\}$ on S^1. To explicitly determine their ordering, we choose $k \in \mathbb{Z}$ such that $kp - 1$ is divisible by q. That such a k always exists is guaranteed because p, q are relatively prime, and its value can be easily determined using the so-called Euclid's algorithm. Turn to elementary number theory for more detailed discussions. Such a k is not unique, but different choices only differ from each other by a multiple of q and will not affect our discussion. For any fixed choice of k, we conclude that $f^s(x)$ is the $(t+1)$-th point on S^1, where $0 \leq t \leq q-1$ is the (unique) integer such that $ks - t$ is divisible by q.

To study the circle maps with irrational rotation numbers, we need to introduce a new (scary-looking) concept.

Let $f : S^1 \to S^1$ be a circle map, then

$$\omega(x) = \bigcap_{n \geq 1} \overline{\{f^i(x) \mid i \geq n\}} \tag{3.1}$$

ω-limit set

is called the *ω-limit set* of x (under the action of f). Here the overhead bar denotes the closure, namely we throw in not only all the points $f^n(x), f^{n+1}(x), \dots$ but also their boundary points. Roughly speaking, $y \in \omega(x)$ simply means that it is the limiting point of some subsequence of this orbit.

Theorem 3.3. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism. Assume that $\rho(f) \notin \mathbb{Q}$. Then*

1. *$\omega(x)$ is independent of the choice of x;*
2. *$\omega(x)$ is either S^1 or a Cantor set.*

From Theorem 3.3, we see that a circle map with irrational rotation number behaves quite similar to an irrational rotation. However, sometimes it behaves essentially the same, namely with $\omega(x) = S^1$, but sometimes there is a slight difference, namely with $\omega(x)$ not dense but a Cantor set. To distinguish these two subtle cases, we quote the Poincaré classification theorem.

Theorem 3.4 (Poincaré Classification Theorem)**.** *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism. Assume that $\rho(f) \notin \mathbb{Q}$. Then*

(1) *there is a continuous monotone map* $h : S^1 \to S^1$ *such that* $h \circ f = R_{\rho(f)} \circ h$, *i.e. we have the following commutative diagram*

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, "h"'] & S^1 \arrow[d, "h"'] \\ S^1 \arrow[r, "R_\rho"] & S^1 \end{tikzcd}$$

(2) *if* $\omega(x) = S^1$ *for some (any)* $x \in \mathbb{R}$, *then* h *is in fact a homeomorphism, and in this case we say that* h conjugates f *to* R_ρ*;* conjugates

(3) *if* $\omega(x)$ *is a Cantor set for some (any)* $x \in \mathbb{R}$, *then* h *is noninvertible, and in this case we say that* R_ρ *is a* factor *of* f *via* h. factor

4. HOMEWORK

In this class, we have learned to

- determine the lifts of circle maps and compute their degrees;
- compute the rotation numbers of circle maps;
- understand the different dynamical behaviors of circle maps with rational and irrational rotation numbers.

Today's homework is Exercises 4.3.1, 4.3.2, 4.3.4 and the following exercise.

Exercise 4.1. Let f be an *orientation-reversing* homeomorphism of S^1. Then are Theorems 3.1, 3.3, 3.4 still valid? If not, which modifications should be made to these theorem? In either case, justify your claim.