LECTURE 15. PRESERVATION OF PHASE VOLUME AND RECURRENCE

DYNAMICAL SYSTEMS (110.421)
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More precisely, consider a dynamical system $f : X \to X$, then a point $x \in X$ (or its orbit) is called *recurrent* if there exists a subsequence $n_1 < n_2 < \dots$ such that $\lim_{k\to\infty} f^{n_k}(x) = x$.

recurrent

1. Preservation of Phase Volume

Let X be a metric space, then a dynamical system $f : X \to X$ is called *volume preserving* or *incompressible* if, for every $x \in X$ and every neighborhood U_x of x, we have $\mathrm{Vol}(U_x) = \mathrm{Vol}(f(U_x))$.

volume preserving
incompressible

Remark 1.1. In the study of classical mechanics, X is usually the phase space of some mechanical system, so a volume preserving map is also called *phase volume preserving*.

phase volume preserving

Theorem 1.2. *Let $X \subseteq \mathbb{R}^n$ be an open set, then a differentiable map $f : X \to \mathbb{R}^n$ preserves volume if and only if $|\operatorname{Jac}(f)| = 1$, where*

$$\operatorname{Jac}(f) = \frac{\partial(u_1,\dots,u_n)}{\partial(x_1,\dots,x_n)} = \begin{vmatrix} \frac{u_1}{x_1} & \cdots & \frac{u_1}{x_n} \\ \vdots & \ddots & \vdots \\ \frac{u_n}{x_1} & \cdots & \frac{u_n}{x_n} \end{vmatrix} \tag{1.1}$$

is the Jacobian *of $f(x_1,\dots,x_n) = (u_1,\dots,u_n)$*

Jacobian

Theorem 1.2 deals with volume preserving maps for discrete dynamical systems, and now let's consider the continuous case.

Theorem 1.3. *Consider the continuous dynamical system $\dot{x} = f(x)$ in $\mathbb{R}^n$. Assume that*

$$\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial f_n}{\partial x_n} \tag{1.2}$$

is identically zero. Then the flow preserves the phase volume.

Proof. Since the infinitesimal volume element is given by $\mathrm{d}V = \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$, by the product rule we have

$$\begin{aligned}
\frac{\mathrm{d}(\mathrm{d}V)}{\mathrm{d}t} &= \frac{\mathrm{d}(\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n)}{\mathrm{d}t} = \sum_{i=1}^{n} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}\dot{x}_i \wedge \dots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^{n} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}f_i(x) \wedge \dots \wedge \mathrm{d}x_n = \sum_{i=1}^{n} \mathrm{d}x_1 \wedge \dots \wedge \left(\sum_{j=1}^{n} \frac{\mathrm{d}f_i}{\mathrm{d}x_j}\,\mathrm{d}x_j \right) \wedge \dots \wedge \mathrm{d}x_n \\
&= \operatorname{div}(f)\,\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n = \operatorname{div}(f)\,\mathrm{d}V.
\end{aligned}$$

Now by assumption $\mathrm{div}(f) = 0$. Hence $\mathrm{d}V$ does not change with time, namely the flow is incompressible. □

2. Recurrence Orbits

Theorem 2.1. *Let X be a closed domain of finite volume in $\mathbb{R}^n$ or $\mathbb{T}^n$, and let $f : X \to X$ be an invertible volume-preserving map. Then the set of recurrent points for f is dense in X.*

uniformly recurrent

Let X be a metric space and $f : X \to X$ a continuous map. Then a point $x \in X$ is called *uniformly recurrent* if, for every $\delta > 0$, the set $\{n \mid d(x, f^n(x)) < \delta\}$ is syndetic [1], namely if there exists $N = N(\delta) \geq 1$ such that for every $n \in \mathbb{Z}$ we always have

$$\min\{d(x, f^{n+k}(x)) \mid 1 \leq k \leq N\} < \delta. \tag{2.1}$$

Example 2.2. Consider the irrational circle rotation $R_\alpha : S^1 \to S^1$, then by the Pigeon Hole Principle every point is uniformly recurrent. In fact, for every $\delta > 0$, we may choose in the above definition $N = [\frac{1}{\delta}] + 1$.

Theorem 2.3. *Let X be compact, and let $f : X \to X$ be a homeomorphism. Then $x \in X$ is uniformly recurrent if and only if the closure of its orbit is a compact minimal set.*

Corollary 2.4. *The orbit closures define a partition of X by compact sets if and only if every point is uniformly recurrent.*

3. Homework

In this class, we have learned to

- determine the volume preservation of maps;
- determine the recurrence and uniform recurrence of points via definitions;
- apply the basic properties of recurrent and uniformly recurrent points.

Today's homework is Exercises 6.1.2, 6.1.4, 6.1.6.

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[1] A subset $S \subseteq \mathbb{Z}$ is called *syndetic* if there exists $N \in \mathbb{N}$ such that $\{n+k \mid 1 \leq k \leq N\} \cap S \neq \emptyset$ for every $n \in \mathbb{Z}$.