LECTURE 20. CHAOS

DYNAMICAL SYSTEMS (110.421)
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1. Topological Transitivity

Proposition 1.1. *Let X be a domain of either some $\mathbb{R}^n$ or $\mathbb{T}^n$ without isolated points, and $f : X \to X$ a continuous map. Then f has a dense orbit in X if and only if, given any nonempty open subsets $U, V \subseteq X$ we have $f^N(U) \cap V \neq \emptyset$ for some $N \geq 1$. In this case, f is called* topologically transitive.

topologically transitive

A continuous map $f : X \to X$ is called *topologically mixing* if, given any nonempty open subsets $U, V \subseteq X$, there exists some $N \geq 1$ such that

topologically mixing

$$f^n(U) \cap V \neq \emptyset \qquad (n > N). \tag{1.1}$$

Proposition 1.2. *Every topologically mixing map is topologically transitive.*

Example 1.3. Expanding maps of S^1 are topologically mixing. In fact, let $f : S^1 \to S^1$ be an expanding map and $F : \mathbb{R} \to \mathbb{R}$ its lift, then $|F'(x)| \geq \lambda$ for some $\lambda > 1$. Now let $U \subseteq S^1$ be a nonempty open subset, and let $I \subseteq \mathbb{R}$ whose image in S^1 is contained in U, then $\ell(f^n(I)) = \min\{\lambda^n \ell(I), 1\}$; in particular, if n is sufficiently large, then $\ell(f^n(I)) = 1$, namely $f^n(I) = S^1$.

Example 1.4. The linear map

$$f : \mathbb{T}^2 \to \mathbb{T}^2, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is topologically mixing. In fact, we note that the eigenvalue $\lambda = \frac{3+\sqrt{5}}{2}$ corresponds to the expanding lines $y = \frac{\sqrt{5}-1}{2}x + C$. Now let $U \subseteq \mathbb{T}^2$ be a nonempty open subset, say containing a segment of the expanding line $y = \frac{\sqrt{5}-1}{2}x + C_0$ of length L_0, and let $\epsilon > 0$. The linear flow $T_{(1, \frac{\sqrt{5}-1}{2})}$ has an irrational slope, so its orbit is dense in $\mathbb{T}^2$; in particular, for every $x \in \mathbb{R}$ there exists an inferior length $L(x)$ such that the projection to $\mathbb{T}^2$ of the line segment on the line $y = \frac{\sqrt{5}-1}{2} + C_0$ starting from x with length $L(x)$ intersects with every open ball on $\mathbb{T}^2$ of radius ϵ (note that we only need to consider only finitely many such balls). Obviously $L(x)$ is continuous and periodic, so it attains a minimal value L. Now we choose n sufficiently large such that $\lambda^n L_0 > L$, then $f^n(U)$ intersects with every open ball on $\mathbb{T}^2$ of radius ϵ.

2. Sensitive Dependence

A map $f : X \to X$ is called to exhibit *sensitive dependence* on initial conditions if there exists some $\Delta > 0$, called a *sensitivity constant*, such that, for every $x \in X$ and every $\epsilon > 0$, there exists some point $y \in X$ and some $N \geq 1$ such that

sensitive dependence
sensitivity constant

$$d(x, y) < \epsilon, \qquad d(f^N(x), f^N(y)) \geq \Delta. \tag{2.1}$$

Roughly speaking, sensitive dependence means that, during the evolution of this dynamical system, even a slightest microscopic error in any initial condition may (and may not) lead to a macroscopic discrepancy.

Proposition 2.1. *Every topologically mixing map has sensitively dependence on initial value conditions, unless the space has only one single point.*

Proof. Let $f : X \to X$ be topologically mixing and $|X| > 1$. Pick up two distinct points $x_1, x_2 \in X$ and set $\Delta = \frac{1}{4}d(x_1, x_2) > 0$. Now for every ball $B(x, \epsilon) \subseteq X$, by assumption for $n \gg 1$ we have

$$f^n(B(x,\epsilon)) \cap B(x_1, \Delta) \neq \emptyset, \qquad f^n(B(x,\epsilon)) \cap B(x_2, \Delta) \neq \emptyset,$$

so there exists $y_1, y_2 \in B(x, \epsilon)$ with $f^n(y_i) \in B(x_i, \Delta)$, whence

$$d(f^n(y_1), f^n(y_2)) \geq d(x_1, x_2) - d(x_1, f^n(y_1)) - d(x_2, f^n(y_2)) > 4\Delta - \Delta - \Delta = 2\Delta.$$

Hence either $d(f^n(y_1), f^n(x)) > \Delta$ or $d(f^n(y_2), f^n(x)) > \Delta$. □

Exercise 2.2. Can you modify the above proof to show that every topologically transitive map also has sensitive dependence on initial conditions? If not, where does the proof break down?

3. CHAOS

chaotic

Let X be a domain of either some $\mathbb{R}^n$ or $\mathbb{T}^n$ without isolated points. Then a continuous map $f : X \to X$ is called *chaotic* if it is topologically transitive and its periodic points are dense.

Example 3.1. The linear expanding map $E_m : S^1 \to S^1$ is chaotic.

Example 3.2. The linear map in Example 1.4 is chaotic.

Theorem 3.3. *Chaotic maps exhibit sensitive dependence on initial conditions, except when the entire space consists of a single periodic orbit.*

Proof. Assume that X has more than one periodic orbit, and let $f : X \to X$ be a chaotic map. Now fix a choice of $x \in X$ and $\epsilon > 0$.

(1) By assumption, there exists a periodic orbit $\mathcal{O}(y)$, say of prime period ℓ, that does not contain x. Write

$$\Delta = \frac{1}{4}d(x, \mathcal{O}(y)) = \frac{1}{4}\min\{d(x, f^n(y)) \mid 0 \leq n < \ell\} = \frac{1}{4}\min\{d(x, f^n(y)) \mid n \geq 0\} > 0.$$

We may also assume that $\epsilon < \Delta$.

(2) By the denseness of periodic points, there exists a periodic point $x_1 \in B(x, \epsilon)$, say with period m. Since $\epsilon < \Delta < d(x, \mathcal{O}(y))$, we have $\mathcal{O}(x_1) \cap \mathcal{O}(y) = \emptyset$. Write

$$V = \bigcap_{i=0}^{m} f^{-i}(B(f^i(y), \Delta)).$$

This is an open neighborhood of y containing the points whose first m iterates track those of y up to Δ.

(3) By the topological transitivity of f, we have $f^k(B(x, \epsilon)) \cap V \neq \emptyset$ for some $k \geq 0$, so there exists $x_2 \in B(x, \epsilon)$ with $f^k(x_2) \in V$. Now let $N = m\lceil \frac{k}{m} \rceil$, the smallest multiple of m that is larger than k, then $0 \leq N - k < m$, so by definition

$$f^N(x_2) = f^{N-k}(f^k(x_2)) \in f^{N-k}(V) \in B(f^{N-k}(y), \Delta), \quad d(f^N(x_2), f^{N-k}(y)) < \Delta.$$

(4) Now by the triangle inequality, and noting that N is a multiple of the period of x_1, we have

$$\begin{aligned} &d(f^N(x), f^N(x_1)) + d(f^N(x), f^N(x_2)) \\ &\qquad \geq d(f^N(x_1), f^N(x_2)) = d(x_1, f^N(x_2)) \geq d(x, f^N(x_2)) - d(x, x_1) \\ &\qquad \geq d(x, f^{N-k}(y)) - d(f^{N-k}(y), f^N(x_2)) - d(x, x_1) \\ &\qquad \geq d(x, \mathcal{O}(y)) - d(f^{N-k}(y), f^N(x_2)) - d(x, x_1) \\ &\qquad \geq 4\Delta - \Delta - \epsilon > 2\Delta. \qquad \square \end{aligned}$$

Hence either $d(f^N(x), f^N(x_1)) > \Delta$ or $d(f^N(x), f^N(x_2)) > \Delta$.

4. Homework

In this class, we have learned to

- apply basic properties of topological transitivity and mixing;
- apply basic properties of chaos;
- apply basic properties of sensitive dependence on initial conditions.

Today's homework is Exercises 7.2.2, 7.2.3 and the following exercise.

Exercise 4.1. Prove or disprove that a contracting map is not topologically mixing. Is it topologically transitive? Verify your claim.