LECTURE 23. ENTROPY

DYNAMICAL SYSTEMS (110.421)
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1. Entropy

Let (X, d) be a compact metric space, and $f : X \to X$ a continuous map. For every $n \geq 1$ we define a new metric

$$d_n^f : X \times X \to \mathbb{R}_+, \qquad (x, y) \mapsto \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)). \tag{1.1}$$

Let $\epsilon > 0$, then a set $E \subseteq X$ is called *ϵ-dense* with respect to d_n^f if

ϵ-dense

$$X = \bigcup_{x \in E} B_{d_n^f}(x, \epsilon) = \bigcup_{x \in E} \{x' \in X \mid d_n^f(x, x') < \epsilon\}, \tag{1.2}$$

and

$$S_n^f(\epsilon) = \min\{|E| \mid E \subseteq X \text{ is } \epsilon\text{-dense with respect to } d_n^f\} \tag{1.3}$$

is called the *ϵ-capacity* of d_n^f. This is the minimal number of the given initial conditions such that, up to time n and precision ϵ, every initial condition is indistinguishable from one of the given ones.

ϵ-capacity

Let $\epsilon > 0$, then a set $E \subseteq X$ is called *ϵ-separated* with respect to d_n^f if $d_n^f(x, x') > \epsilon$ for every distinct points $x, x' \in E$, and we write

ϵ-separated

$$N_n^f(\epsilon) = \max\{|E| \mid E \subseteq X \text{ is } \epsilon\text{-separated with respect to } d_n^f\}. \tag{1.4}$$

This is the maximal number of the length-n orbits that can be recognized with precision ϵ.

Proposition 1.1. *We have*

$$\lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log S_n^f(\epsilon)}{n} = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log N_n^f(\epsilon)}{n}. \tag{1.5}$$

Let (X, d) be a compact metric space, and $f : X \to X$ a continuous map. Then the quantity

$$h_{top}(f) = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log S_n^f(\epsilon)}{n} = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log N_n^f(\epsilon)}{n} \tag{1.6}$$

is called the *topological entropy* of X, or the *entropy* of X for short.

topological entropy
entropy

Proposition 1.2. *Topological entropy is an invariant of topological conjugacy.*

Proof. Let d, d' be equivalent metrics on X, then for every ϵ there exists $\delta = \delta(\epsilon) > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$ for every $x \in X$, whence

$$S_n^f(\epsilon, d) \leq S_n^f(\delta, d'), \qquad \overline{\lim_{n \to \infty}} \frac{\log S_n^f(\epsilon, d)}{n} \leq \overline{\lim_{n \to \infty}} \frac{\log S_n^f(\delta, d')}{n}.$$

Obviously $\lim_{\epsilon\to 0}\delta = 0$, so we have

$$h_{top,d}(f) = \lim_{\epsilon\to 0}\overline{\lim_{n\to\infty}}\frac{\log S_n^f(\epsilon, d)}{n} \leq \lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\frac{\log S_n^f(\delta, d')}{n} = h_{top,d'}(f).$$

Similarly, we can show that $h_{top,d'}(f) \leq h_{top,d}(f)$. Hence $h_{top,d}(f) = h_{top,d'}(f)$, i.e. the topological entropy is determined by the equivalence class of the metric.

Now let $f : (X, d_X) \to (X, d_X)$ and $g : (Y, d_Y) \to (Y, d_Y)$ be topologically conjugate via a homeomorphism $h : X \to Y$, then the pull-back metric of d_X on Y

$$h^* d_X(y_1, y_2) = d_X(h^{-1}(y_1), h^{-1}(y_2))$$

is equivalent to d_Y. Now $h : (X, d_X) \to (Y, h^* d_X)$ is an isometry, so

$$h_{top,d_X}(f) = h_{top,h^* d_X}(g) = h_{top,d_Y}(g). \qquad \square$$

2. EXAMPLES

Example 2.1. Let X be a compact metric space and $f : X \to X$ a contracting map, then $h_{top}(f) = 0$.

Example 2.2. Let X be a compact metric space and $f : X \to X$ an isometry, then $h_{top}(f) = 0$.

Example 2.3. Let

$$f : S^1 \times [0, 1] \to S^1 \times [0, 1], \qquad ([x], y) \mapsto ([x + y], y)$$

be the additive twist. Then we have $h_{top}(f) = 0$. In fact, it is easy to see that a minimal ϵ-dense subset with respect to d_n^f consists of $\frac{n}{\epsilon^2}$ points, with $\frac{1}{\epsilon}$ points horizontally evenly distributed, and $\frac{n}{\epsilon}$ points vertically evenly distributed. Hence

$$S_n^f(\epsilon) = \frac{n}{\epsilon^2}, \qquad h_{top}(f) = 0.$$

Example 2.4. Let $f : S^1 \to S^1$ be an expanding map of degree m, then

$$h_{top}(f) = h_{top}(E_m) = \log |m|. \tag{2.1}$$

3. HOMEWORK

In this class, we have learned to

- understand and apply the basic properties of the topological entropy;
- compute the topological entropies for simple dynamical systems.

There is no homework for today.