12. \( V \) is an n-dimensional vector space over \( F \).
   \( T: V \rightarrow V \) is linear transformation.
   Range (\( T \)) and nullspace (\( T \)) are identical.
   Prove that \( n \) is even.

   By Thm 2 on p.71, \( n = \dim V = \text{rank } T + \text{nullity } T \).
   Since range (\( T \)) = nullspace (\( T \)), their dimensions are equal, i.e., \( \text{rank } T = \text{nullity } T = m \geq 0 \) (integer).

   Thus, \( n = \dim V = m + m = 2m \), so \( n \) is even. \( \checkmark \)

13. \( V \) is a vector space.
   \( T: V \rightarrow V \) is linear transformation.
   Prove the following 2 statements are equivalent:
   (a) range (\( T \)) \( \cap \) nullspace (\( T \)) = \( \{0\} \)
   (b) If \( T(Tx) = 0 \), then \( Tx = 0 \).

   \( (a) \Rightarrow (b) \): Suppose range (\( T \)) \( \cap \) nullspace (\( T \)) = \( \{0\} \)
   and \( T(Tx) = 0 \). Show \( Tx = 0 \).
   \[ T(Tx) = 0 \Rightarrow Tx \in \text{nullspace}(T) \]
   Also, \( Tx \in \text{Range}(T) \)
   So, \( Tx \in \text{Range}(T) \) \( \cap \) nullspace (\( T \))
   Thus, \( Tx = 0 \). \( \checkmark \)

   \( (b) \Rightarrow (a) \): Suppose \( T(Tx) = 0 \Rightarrow Tx = 0 \).
   Show range (\( T \)) \( \cap \) nullspace (\( T \)) = \( \{0\} \).
   The given info says that whenever an element is in the range and the kernel, it must be 0.
   Thus, range (\( T \)) \( \cap \) nullspace (\( T \)) = \( \{0\} \). \( \checkmark \)
11. \( V \) - finite dimensional vector space

\[ T : V \to V \]

Suppose \( \text{rank}(T) = \text{rank}(T^2) \).

Prove: \( \text{range}(T) \cap \text{nullspace}(T) = \{0\} \).

By #13 on p.74, proving the above is equivalent to proving \( T(T\alpha) = 0 \Rightarrow T\alpha = 0 \), so let's prove the latter.

Since \( \text{rank}(T) = \text{rank}(T^2) \), by the Rank-Nullity Thm,

\[ \text{nullity}(T) = \text{nullity}(T^2) \]

(Recall that nullity is the dimension of the kernel.)

Now, \( \text{kernel}(T) \leq \text{kernel}(T^2) \).

Combining the last 2 statements, \( \text{kernel}(T) = \text{kernel}(T^2) \).

Thus, if \( \alpha \in \text{kernel}(T^2) \), i.e. \( T(T\alpha) = T^2\alpha = 0 \),

then \( \alpha \in \text{kernel}(T) \), i.e. \( T\alpha = 0 \). \( \checkmark \)

---

7. \( V, W \) - vector spaces over \( F \)

\( U : V \to W \) an isomorphism

Prove that \( T \mapsto UTU^{-1} \) is an isomorphism of \( L(V,V) \) onto \( L(W,W) \).

\[ T \mapsto UTU^{-1} \]

Method 1:

Show \( T \) is 1-1 by showing \( \text{ker}(T) = 0 \):

Suppose \( T(T\alpha) = UTU^{-1} = 0 \)

\( UTU^{-1} \) are isomorphisms and therefore have full rank, so \( T \) must be 1-1. \( \checkmark \)

Show \( T \) is onto: Let \( S \in L(W,W) \).

I need to show that \( S \) can be mapped to.

Consider \( T(U^{-1}SU) \), where \( U^{-1}SU \in L(V,V) \)

\[ T(U^{-1}SU) = UTU^{-1}SU = S \]

Hence, \( \text{Range}(T) = L(W,W) \). \( \checkmark \)

Method 2:

Show \( T \) is an invertible linear transformation by exhibiting its inverse.

\[ \varphi^{-1}(S) = U^{-1}SU \], where \( S \in L(W,W) \).

\[ \varphi^{-1} \circ \varphi(U) = U^{-1}(UTU^{-1}) \]

To show this is indeed the inverse, 

Likewise, \( \varphi \circ \varphi^{-1}(S) = \text{the identity on } L(W,W) \).
Let $T$ be the linear operator on $\mathbb{R}^3$, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$. Find a basis for the range of $T$ and the kernel of $T$.

Let's row-reduce:

$$
A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R
$$

$Ax = 0$ and $Rx = 0$ have the same solutions.

$$
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$x_1 - x_3 = 0$

$x_2 + x_3 = 0$

Let $x_3 = c$, so solutions look like $(c, -c, c)$. Thus, a basis for the kernel of $T$ is $\{(1, -1, 1)\}$.

A basis for the range of $T$ consists of the linearly independent columns of $A$.

From our row-reduced matrix, we see that the 3rd column is a linear combination of the 1st two.

Thus, a basis for the range of $T$ is $\{(1, 0, -1), (2, 1, 3)\}$.

---

W: space of all $n \times 1$ column matrices over $F$

$A$: $n \times n$ matrix over $F$

$A$ defines a linear operator $L_A$ on $W$ through left multiplication: $L_A(x) = Ax$.

Prove that every linear operator on $W$ is left multiplication by some $n \times n$ matrix, i.e., is $L_A$ for some $A$.

Let $T$ be a linear operator on $W$. By Thm II, $T$ can be represented by a matrix $A$, where $T(x) = Ax$. Then $T$ is left multiplication by $A$, i.e., $T = L_A$. 

---
Now suppose $V$ is an $n$-dimensional vector space over $F$, and let $B$ be an ordered basis for $V$. For each $\alpha \in V$, define $U_\alpha = [\alpha]_B^\uparrow$. Prove that $U$ is an isomorphism of $V$ onto $W$.

Show $U$ is 1-1:

Let $U_\alpha = [\alpha]_B^\uparrow = 0$.

Let's also suppose $B = \{v_1, \ldots, v_n\}$. Thus, $\alpha = a_1v_1 + \ldots + a_nv_n$ for some $a_i, i = 1, \ldots, n$.

Rewriting, $U_\alpha = U(a_1v_1 + \ldots + a_nv_n) = [\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}] = 0$. Thus, $a_1 = \ldots = a_n = 0 \Rightarrow \alpha = 0$.

This means $\ker(U)$ is 0, and hence $U$ is 1-1.

Show $U$ is onto:

Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in W$. Is there an $\alpha \in V$ such that $U_\alpha = X$?

Yes. Let $\alpha = x_1v_1 + \ldots + x_nv_n$. (Note that $\alpha \in V$)

Then $U_\alpha = [\alpha]_B^\uparrow = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Since $X$ is a generic element of $W$, $U$ is onto, i.e. $\text{range}(U) = W$.

Thus $U$ is an isomorphism.

12. $V$ is an $n$-dimensional vector space over $F$.

$B = \{\alpha_1, \ldots, \alpha_n\}$ is an ordered basis for $V$.

(a) According to Thm 1, there is a unique linear operator $T$ on $V$ such that

$T\alpha_j = \alpha_{j+1}$, $j = 1, \ldots, n-1$

$T\alpha_n = 0$.

What is the matrix of $T$ in the ordered basis $B$?
(a) continued

\[

t_1(x_1) = x_2 = 0x_1 + 1x_2 + 0x_3 + \ldots + 0x_n \\
t_1(x_2) = x_3 = 0x_1 + 0x_2 + 1x_3 + 0x_4 + \ldots + 0x_n \\
\vdots \\
t_1(x_n) = 0 = 0x_1 + \ldots + 0x_n \\
\]

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

(b) Prove that \( T^n = 0 \), but \( T^{n-1} \neq 0 \).

(I need to show \( T^n \) is the zero map, but \( T^{n-1} \) is not.)

\[
T^2 x_j = x_{j+2} \quad j = 1, \ldots, n-2
\]

\[
T^2 x_{n-1} = T(T x_{n-1}) = T(x_n) = 0
\]

\[
T^2 x_n = T(T x_n) = T(0) = 0
\]

\[
T^3 x_j = x_{j+3} \quad j = 1, \ldots, n-3
\]

\[
T^3 x_{n-2} = T(T x_{n-2}) = T^2(x_{n-1}) = 0
\]

\[
T^3 x_{n-1} = T(T^2 x_{n-1}) = T(0) = 0
\]

\[
T^3 x_n = T^2 (T x_n) = T^2(0) = 0
\]

\[
\vdots
\]

\[
T^{n-1} x_j = x_{j+n-1} \quad j = 1
\]

\[
T^{n-1} x_2 = T(T^{n-2} x_2) = T(x_n) = 0
\]

Thus, \( T^{n-1} \) is not the zero map.

\[
T^n x_j = T(T^{n-1} x_j) = T(0) = 0
\]

for \( j = 2, \ldots, n \).

Thus, \( T^n \) is the zero map.
Let $S$ be any linear operator on $V$ such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $B'$ for $V$ such that the matrix of $S$ in the ordered basis $B'$ is the matrix $A$ of part (a).

We will show such a basis exists by constructing it.

Let $v \in V$ such that $S^{n-1}v \neq 0$. We know such a vector exists since $S^{n-1}$ is not the zero map.

Claim: \( \{v, Sv, \ldots, S^{n-1}v\} \) is a basis for $V$.

Since $V$ is finite dimensional, it suffices to show that these vectors are linearly independent.

Suppose \( c_0v + c_1Sv + \ldots + c_{n-1}S^{n-1}v = 0 \).

Then \( S^{n-1}(c_0v + c_1Sv + \ldots + c_{n-1}S^{n-1}v) = 0 \) also.

\[ S^{n-1}(c_0v) = 0 + \ldots + 0 \]

So, \( c_0S^{n-1}v = 0 \), but \( S^{n-1}v \neq 0 \) by above assumption, so \( c_0 \) is 0.

Now, consider \( c_1Sv + \ldots + c_{n-1}S^{n-1}v = 0 \).

Then \( S^{n-2}(c_1Sv + \ldots + c_{n-1}S^{n-1}v) = 0 \) also.

\[ S^{n-2}(c_1Sv) = 0 + \ldots + 0 \]

So, \( c_1S^{n-2}(Sv) = 0 \)

\[ c_1S^{n-1}v = 0 \), but \( S^{n-1}v \neq 0 \), so \( c_1 = 0 \).

Continuing in the same manner, \( c_0 = c_1 = c_2 = \ldots = c_{n-1} = 0 \), so \( \{v, Sv, \ldots, S^{n-1}v\} \) is a lin. ind. set, and hence a basis for $V$.

Note that \( \{v, Sv, \ldots, S^{n-1}v\} \) satisfies the properties in part (a).

Let $B' = \{v, Sv, \ldots, S^{n-1}v\}$.

Then the matrix of $S$ in the ordered basis $B'$ is the same matrix as that in part (a).

Prove that if $M$ and $N$ are $n \times n$ matrices over $F$ such that $M^n = N^n = 0$, but $M^{n-1} \neq 0$ and $N^{n-1} \neq 0$, then $M$ and $N$ are similar.

Just apply part (c).