These notes are incomplete – they will be updated regularly.

LIE GROUPS, LIE ALGEBRAS, AND REPRESENTATIONS
SPRING SEMESTER – 2008

RICHARD A. WENTWORTH

Contents

1. Lie groups and Lie algebras 2
   1.1. Definition and examples of Lie groups 2
   1.2. Left invariant vector fields and Lie algebras 3
   1.3. The exponential map 6
2. Representations of Lie groups and Lie algebras 9
   2.1. Definitions and examples 9
   2.2. The adjoint representation 9
   2.3. Semisimplicity and Schur’s lemma 9
   2.4. Haar measure and Weyl’s trick 9
   2.5. Characters 9
   2.6. Examples 9
   2.7. Peter-Weyl theorem 9
3. Maximal tori 9
   3.1. Existence 9
   3.2. Weyl group 9
   3.3. Cartan subalgebras 9
4. Classification of complex semisimple Lie algebras 9
   4.1. Roots 9
   4.2. Dynkin diagrams 9
   4.3. Stiefel diagrams 9
   4.4. Classification 9
5. Weyl character formula 9
   5.1. Weights and multiplicities 9
   5.2. Examples 9

... without fantasy one would never become a mathematician, and what gave me a place among the mathematicians of our day, despite my lack of knowledge and form, was the audacity of my thinking. - Sophus Lie

Date: February 4, 2008 (last revision).
1. Lie groups and Lie algebras

1.1. Definition and examples of Lie groups.

**Definition 1.1.** A (real, complex) Lie group $G$ is a group that has the structure of a (real, complex) differentiable manifold so that the multiplication and inverse maps are smooth.

Some examples:

- The simplest Lie group is perhaps $\mathbb{R}$ with its additive structure. The simplest compact example is the circle $U(1) = \mathbb{R}/\mathbb{Z}$. More generally, take finite dimensional vector spaces such as $\mathbb{R}^n$, and those vector spaces divided by lattices $\mathbb{R}^n/\Gamma$, $\Gamma \cong \mathbb{Z}^n$.

- Let $V$ be an $n$-dimensional vector space over $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Then $GL(V) = \{\text{invertible endomorphisms } A : V \to V\}$ Note that invertibility is equivalent to the non-vanishing of $\det A$, which is smooth (in fact, polynomial) in the entries of $A$. Hence, $GL(V)$ is an open submanifold of the space of all endomorphisms $\text{End}(V)$. The manifold structure on the latter comes from the identification with the vector space of matrix entries. We will mostly denote these groups $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, and $GL(n, \mathbb{H})$.

- Let $SL(V) \subset GL(V)$ denote the subgroup with $\det A = 1$. To see the manifold structure, by the implicit function theorem we need to show that 1 is a regular value for the determinant map. By an important formula, at a point $A$ with $\det A = 1$,

\[ D \det(A) X = D \log \det(A) X = \text{Tr}(A^{-1}X) \]

and this is clearly surjective, since it is not identically zero.

- Let $O(n) \subset GL(n, \mathbb{R})$ be the group of length preserving matrices. This is the *orthogonal group*. We have

\[ O(n) = \{A \in GL(n, \mathbb{R}) : AA^T = I\} \]

To see the manifold structure, we need to show that zero is a regular value for the map $F : GL(n, \mathbb{R}) \to \text{Sym}(n, \mathbb{R})$ defined by $F(A) = AA^T - I$, where $\text{Sym}(n, \mathbb{R})$ are the symmetric matrices. We have

\[ DF(A)X = XA^T + AX^T \]

and this is surjective, for if $Y$ is symmetric, $A \in O(n)$, and $X = YA/2$, then $DF(A)X = Y$.

- In a similar way, one defines the *unitary* and *symplectic* groups $U(n)$ and $\text{Sp}(n)$. More precisely, these are subgroups of $GL(n, \mathbb{C})$ and $GL(n, \mathbb{H})$ preserving hermitian and quaternionic norms. E.g.

\[ U(n) = \{A \in GL(n, \mathbb{C}) : AA^* = I\} \]

where $A^* = \overline{A}^T$.

- We can also define the groups $SO(n)$ and $SU(n)$ by the requirement $\det A = 1$.

Notice the following:
• The real groups $SU(n)$ and $SL(n, \mathbb{R})$ are both subgroups of the complex group $SL(n, \mathbb{C})$.

• $SU(n)$ is compact whereas $SL(n, \mathbb{R})$ is non-compact.

• $SU(n) \cap SL(n, \mathbb{R}) = SO(n)$.

This structure is an important feature that holds in great generality.

For example, suppose $A \in SU(2)$. Then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Since $A^* = A^{-1}$, we have

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in GL(2, \mathbb{C}) : |a|^2 + |b|^2 = 1 \right\}$$

In this way, we see that diffeomorphically $SU(2) \simeq S^3$. Notice that $U(1) \subset SU(2)$ by the requirement $b = 0$. The quotient space $SU(2)/U(1) \simeq S^2$, and this is known as the Hopf fibration. We can see this as follows. $SU(2)$ acts on $S^2 = \mathbb{P}^1$ by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} [z_1, z_2] = [az_1 + bz_2, -\bar{b}z_1 + \bar{a}z_2]$$

This action is clearly transitive, and the stabilizer of $[1, 0]$ is precisely the $U(1)$ above. This realizes $\mathbb{P}^1$ as a homogeneous space.

The requirement that $a, b \in \mathbb{R}$ implies

$$SU(2) \cap SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2, \mathbb{R}) : a^2 + b^2 = 1 \right\} \simeq U(1)$$

On the other hand, $SL(2, \mathbb{R})$ acts on the upper half space $\mathbb{H}^2$ (not the quaternions!) by linear fractional transformations:

$$\tau = (x, y) \mapsto A\tau = \frac{a\tau + b}{c\tau + d}$$

It is easy to see that this is a transitive action, and that the stabilizer of $\tau = i$, say, is the $U(1)$ above. In this way we see that $SL(2, \mathbb{R})/U(1)$ is diffeomorphic to the upper half plane, which is an open set in $S^2 = SU(2)/U(1)$. This is an example of the Borel embedding.

1.2. Left invariant vector fields and Lie algebras.

**Definition 1.2.** Let $G$ be a connected Lie group, $\mathfrak{g} = T_eG$. Then $\mathfrak{g}$ is called the Lie algebra of $G$.

An important fact is that the tangent bundle of a Lie group is trivial $TG \simeq G \times \mathfrak{g}$. A trivialization is given by the following. Any $g \in G$ gives rise to two diffeomorphisms of $G$ by multiplication on the left or on the right.

$$l_g : G \to G : l_g(h) = gh, \quad r_g : G \to G : r_g(h) = hg$$

Now the derivative $Dl_g : T_hG \to T_{gh}G$. We make the following

**Definition 1.3.** A vector field $X_g$ on $G$ is called left invariant if $Dl_g(X_h) = X_{gh}$ for all $g, h \in G$. 
The space of left invariant vector fields is isomorphic to \( \mathfrak{g} \) by setting \( X_g = Dl_g(X) \), for \( X \in \mathfrak{g} \). This gives the desired trivialization. We will use the abbreviation l.i.v.f. for “left invariant vector field”.

**Remark 1.4.** For a matrix group such as the ones we have considered above, \( Dl_g \) is simply multiplication by \( g \). Hence, \( X_g = gX \).

For any smooth manifold \( M \), there is a product on the space of vector fields \( \mathfrak{X}(M) \) on \( M \) defined as follows. For \( X, Y \in \mathfrak{X}(M) \) and a function \( f \) on \( M \), let

\[
[X, Y](f) = X(Y(f)) - Y(X(f))
\]

To be explicit, in local coordinates \( e^i \), write

\[
X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial e^i} \quad Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial e^i}
\]

Then in these coordinates

\[
[X, Y] = \sum_{i,j=1}^{n} \left\{ X^i \frac{\partial Y^j}{\partial e^i} - Y^i \frac{\partial X^j}{\partial e^i} \right\} \frac{\partial}{\partial e^j}
\]

The bracket satisfies

1. \([X, Y] = -[Y, X]\)
2. \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \) (Jacobi identity).

Given a Lie group, \( \mathfrak{g} = T_eG \) has been identified with the space of l.i.v.f.‘s, and so \( \mathfrak{g} \) inherits the bracket operation. Namely, define \([\ , \ ]\) on \( \mathfrak{g} \) by

\[
[X, Y]_{\mathfrak{g}} = [X_g, Y_g]\bigg|_{g=e}
\]

We now abstract this.

**Definition 1.5.** A finite dimensional real (complex) vector space \( \mathfrak{g} \) with a bilinear multiplication \([\ , \ ]_{\mathfrak{g}}\) satisfying properties (1) and (2) above is called a real (complex) Lie algebra.

Hence, if \( G \) is a Lie group then \( \mathfrak{g} = T_eG \) with bracket (1.4) is a Lie algebra. Conversely, every (finite dimensional) Lie algebra arises in this way.

**Remark 1.6.** If \( \mathfrak{g} \) is a real Lie algebra, the complexification \( \mathfrak{g}^C = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \) is a complex Lie algebra. This is an important construction.

**Proposition 1.7.** For matrix groups, the Lie bracket is given by the commutator of matrix multiplication.

**Proof.** It suffices to prove this for \( GL(n) \). For coordinates we choose the elementary matrices: \( x^{ij}(A) = A^{ij} \). If \( X = \sum_{i,j=1}^{n} X^{ij} \frac{\partial}{\partial x^{ij}} \), then by Remark 1.4, \( X_g = \sum_{i,j=1}^{n} g^{ik} X^{kj} \frac{\partial}{\partial x^{ij}} \). It follows that

\[
\frac{\partial X_g^{pq}}{\partial x^{ij}} \bigg|_{g=e} = \delta^{pq} X^{ij}
\]
Hence, plugging into the expression above,

\[ [X_g, Y_g]_{g=e} = \sum_{i,j,p,q=1}^{n} \{ X_{ij} \frac{\partial Y_{pq}}{\partial x_{ij}} - Y_{ij} \frac{\partial X_{pq}}{\partial x_{ij}} \} \frac{\partial}{\partial x_{pq}} \]

\[ = \sum_{i,j,p,q=1}^{n} \{ X_{ij} \delta_{pi} Y_{jq} - Y_{ij} \delta_{pi} X_{jq} \} \frac{\partial}{\partial x_{pq}} \]

\[ = \sum_{i,p,j,q=1}^{n} \{ X_{pj} Y_{jq} - Y_{pj} X_{jq} \} \frac{\partial}{\partial x_{pq}} \]

\[ = XY - YX \]

□

Examples:

• If \( G \) is abelian, \( g \) is a \textit{trivial} Lie algebra, i.e. all brackets vanish. Indeed, suppose \( G \) is a matrix group. If \( g(t) \) and \( h(s) \) are curves through the identity with \( \dot{g}(0) = X, \dot{h}(0) = Y \), then

\[ \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial s} \bigg|_{s=0} (ghg^{-1}h^{-1}) = \frac{\partial}{\partial t} \bigg|_{t=0} (gYg^{-1} - Y) = XY - YX \]

For the general argument, see Proposition 1.13 below.

• Since \( \text{GL}(V) \) is an open subset of the vector space of endomorphisms, \( \mathfrak{gl}(V) \simeq \text{End}(V) \). By Proposition 1.7, the Lie algebra structure is simply the commutator of matrix multiplication. Suppose \( \dim V = n \). A basis is given by the elementary matrices \( \{ e(i, j) \}_{i,j=1}^{n} \), i.e.

\[ (e(i, j)e(p, q))_{kl} = \sum_{r=1}^{n} (e(i, j))_{kr} (e(p, q))_{rl} = \sum_{l=1}^{n} \delta_{ik} \delta_{rj} \delta_{pr} \delta_{ql} = \delta_{ik} \delta_{jp} \delta_{ql} = \delta_{jp} e(i, q) \]

we have

\[ (1.5) \quad [e(i, j), e(p, q)] = \delta_{jp} e(i, q) - \delta_{qi} e(p, j) \]

• From (1.1) we see that

\[ \mathfrak{sl}(V) = \{ X \in \text{End}(V) : \text{Tr}(X) = 0 \} \]

Notice that any endomorphism can be decomposed into traceless and trace pieces:

\[ X = \left( X - \frac{1}{n} \text{Tr} X \cdot I \right) + \frac{1}{n} \text{Tr} X \cdot I \]

Hence, if \( V \) is a vector space over \( \mathbb{K} \) we have a canonical decomposition of Lie algebras:

\[ \mathfrak{gl}(V) = \mathbb{K} \oplus \mathfrak{sl}(V) \]

• From (1.2) we see that

\[ \mathfrak{o}(n) = \{ X \in \text{End}(\mathbb{R}^n) : X^T = -X \} \]
i.e. the skew-symmetric matrices. Also, \( \mathfrak{so}(n) \) consists of the traceless, skew-symmetric matrices. Similarly,

\[
\mathfrak{u}(n) = \{ X \in \text{End}(\mathbb{C}^n) : X^* = -X \}
\]

or skew-hermitian matrices, and \( \mathfrak{su}(n) \) are traceless, skew-hermitian.

- A basis for \( \mathfrak{su}(2) \) is given by

\[
\sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]

with commutation relations

\[
[\sigma_1, \sigma_2] = -\sigma_3 \quad [\sigma_2, \sigma_3] = -\sigma_1 \quad [\sigma_3, \sigma_1] = -\sigma_2
\]

These (up to scale) are the relations for the cross product on \( \mathbb{R}^3 \).

- For \( \mathfrak{sl}(2, \mathbb{R}) \), take

\[
X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

and one checks the same commutation relations

\[
[X, Y] = 2Y \quad [X, Z] = -2Z \quad [Y, Z] = X
\]

Notice the 2-dimensional subalgebras. It is clear that \( \mathfrak{su}(2) \) does not have 2-dimensional subalgebras, so \( \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathfrak{su}(2) \) are not isomorphic. On the other hand, their complexifications are:

\[
\mathfrak{su}(2)^C \simeq \mathfrak{sl}(2, \mathbb{R})^C \simeq \mathfrak{sl}(2, \mathbb{C})
\]

1.3. The exponential map.

**Definition 1.8.** A one parameter subgroup (1-PS) of a Lie group \( G \) is a homomorphism \( \mathbb{R} \to G \).

A 1-PS is determined by an element of \( \mathfrak{g} \) in the following way. Given \( X \in \mathfrak{g} \), let \( \gamma(t) \) be a solution to the differential equation

\[
\dot{\gamma}(t) = X_{\gamma(t)} \quad \gamma(0) = e
\]

i.e. \( \gamma(t) \) is the flow through the identity of the l.i.v.f. of \( X \).

**Lemma 1.9.** The map \( t \mapsto \gamma(t) \) defines a homomorphism \( \mathbb{R} \to G \). In particular, \( \gamma(-t) = \gamma(t)^{-1} \).

**Proof.** We need to show \( \gamma(s + t) = \gamma(s)\gamma(t) \) for all \( t, s \). Fix \( s \) and let

\[
\sigma(t) = \gamma(s)\gamma(t) \quad \rho(t) = \gamma X(s + t)
\]

Then

\[
\dot{\sigma}(t) = Dl_{\gamma(s)}\dot{\gamma}(t) = Dl_{\gamma(s)}X_{\gamma(t)}
\]

\[
= Dl_{\gamma(s)}Dl_{\gamma(t)}X = Dl_{\gamma(s)\gamma(t)}X
\]

\[
= X_{\sigma(t)}
\]
Hence, \( \sigma(t) \) is the integral curve of the l.i.v.f. of \( X \) through \( \sigma(0) = \gamma(s) \). On the other hand,
\[
\dot{\rho}(t) = \dot{\gamma}(s + t) = X_{\gamma(s + t)} = X_{\rho(t)}
\]
So \( \rho(t) \) is the integral curve of the l.i.v.f. of \( X \) through \( \rho(0) = \gamma(s) \). By the uniqueness of integral curves, these must coincide.

**Lemma 1.10.** Let \( \gamma(t) \) be the flow of \( X_g \) through the identity. Then
\[
[X, Y]_g = \frac{d}{dt} \bigg|_{t=0} Dr_{\gamma(-t)}Y_{\gamma(t)}
\]

**Proof.** Notice that \( \Phi_t(g) = g\gamma(t) \) is the flow of \( X_g \) through the point \( g \). Moreover, \( D\Phi_t Y_g = Dr_{\gamma(t)}Y_g \). Hence, by definition of the Lie derivative
\[
[X, Y]_g = L_X Y_t = \frac{d}{dt} \bigg|_{t=0} D\Phi_{-t}Y_{\gamma(t)} = \frac{d}{dt} \bigg|_{t=0} Dr_{\gamma(-t)}Y_{\gamma(t)}
\]

**Definition 1.11.** The exponential map \( \exp : \mathfrak{g} \to G \) is defined by \( \exp(X) = \gamma(1) \), where \( \gamma(t) \) is the flow of \( X_g \) through the identity.

Notice that \( d\exp(0) \) is the identity. Hence, by the implicit function theorem, \( \exp \) is a local diffeomorphism. Here is the justification for the terminology:

**Claim 1.** Let \( G \) be a matrix group. Then
\[
\exp(X) = \sum_{m=0}^{\infty} \frac{X^m}{m!}
\]

**Proof.** Let \( E(X) \) be the right hand side, and \( \sigma(t) = E(tX) \). Then \( \sigma(0) = I \) and \( \dot{\sigma}(t) = \sigma(t)X = X_{\sigma(t)} \). Hence, \( \sigma(t) \) is the flow of the l.i.v.f. of \( X \), and the claim holds.

Here is one use of the exponential map.

**Proposition 1.12.** Let \( \phi : H \to G \) be a homomorphism of Lie groups with \( H \) connected. The derivative \( D\phi(e) : \mathfrak{h} \to \mathfrak{g} \) is a homomorphism of Lie algebras. Moreover, any other homomorphism with the same derivative is equal to \( f \).

**Proof.** We first show that the derivative \( D\phi : T_hH \to T_{\phi(h)}G \) takes l.i.v.f.’s to l.i.v.f.’s. Indeed, if \( X \in \mathfrak{h} \), \( Y = D\phi(e)X \), we need to show that \( D\phi(h)X_h = Y_{\phi(h)} \). Let \( \gamma(t) \) be a curve in \( H \), \( \gamma(0) = e \), \( \dot{\gamma}(0) = X \). Then
\[
\frac{d}{dt} \bigg|_{t=0} \phi(h\gamma(t)) = D\phi(h)Dl_hX = D\phi(h)X_h
\]
On the other hand,
\[
\frac{d}{dt} \bigg|_{t=0} \phi(h)\phi(\gamma(t)) = Dl_{\phi(h)} \frac{d}{dt} \bigg|_{t=0} \phi(\gamma(t)) = Dl_{\phi(h)}D\phi(e)X = Dl_{\phi(h)}Y = Y_{\phi(h)}
\]
Since $\phi(h \gamma(t)) = \phi(h) \phi(\gamma(t))$, this proves the claim. It follows that

$$D\phi(e)[X, Y]_h = D\phi(h)[X_h, Y_h] \bigg|_{h=e}$$

$$= [D\phi(h)X_h, D\phi(h)Y_h] \bigg|_{h=e}$$

$$= [(D\phi(e)X)_{\phi(h)}, (D\phi(e)Y)_{\phi(h)}] \bigg|_{h=e}$$

$$= [D\phi(e)X, D\phi(e)Y]_g$$

Hence, $D\phi(e)$ is a homomorphism. Since l.i.v.f.'s are sent to l.i.v.f.'s, $\phi$ maps flows of these vector fields to the corresponding flows. The following diagram therefore commutes:

\begin{align*}
    \mathfrak{h} & \xrightarrow{\exp} H \\
    D\phi(e) & \downarrow \quad \quad \downarrow \phi \\
    \mathfrak{g} & \xrightarrow{\exp} G
\end{align*}

Since exp is a diffeomorphism on a neighborhood $U$ about the origin, it follows from this diagram that if $D\tilde{\phi}(e) = D\phi(e)$, then $\tilde{\phi} = \phi$ on $U$. Now consider

$$\tilde{H} = \left\{ h \in H : \tilde{\phi}(h) = \phi(h) \right\}$$

This set is clearly closed and contains $U$. If $h \in \tilde{H}$, then since $\phi$ is a homomorphism, $hU$ is an open neighborhood of $h$ contained in $\tilde{H}$. Hence, $\tilde{H}$ is open and closed and nonempty. It must equal $H$ by connectivity.

\[ \square \]

**Proposition 1.13.** The Lie algebra of an abelian Lie group is trivial. Moreover, the exponential map of an abelian Lie group is a homomorphism.

**Proof.** Fix $X, Y \in \mathfrak{g}$, and let $\gamma(t) = \exp(tX)$. If $G$ is abelian, then $l_g = r_g$ and $Dl_g = Dr_g$. Also, recall from Lemma 1.9 that $\gamma(-t) \gamma(t) = e$. Now by Lemma 1.10 we have

\[
[X, Y]_g = L_{X_g} Y_g \bigg|_{g=e} = \frac{d}{dt} \bigg|_{t=0} D\gamma(-t) Y_{\gamma(t)}
\]

\[
= \frac{d}{dt} \bigg|_{t=0} Dl_{\gamma(-t)} Y_{\gamma(t)}
\]

\[
= \frac{d}{dt} \bigg|_{t=0} Dl_{\gamma(-t)} Dl_{\gamma(t)} Y
\]

\[
= \frac{d}{dt} \bigg|_{t=0} Dl_{\gamma(-t)\gamma(t)} Y
\]

\[
= \frac{d}{dt} \bigg|_{t=0} Dl_e Y = \frac{d}{dt} \bigg|_{t=0} Y = 0
\]
To prove the second statement, notice that the multiplication map \( \mu : G \times G \to G \) is a homomorphism if and only if \( G \) is abelian. Also, \( D\mu : g \times g \to g : (X,Y) \mapsto X + Y \). The result now follows from the commutativity of the diagram (1.6).

2. **Representations of Lie groups and Lie algebras**

2.1. **Definitions and examples.**

2.2. **The adjoint representation.** We may think of one vector field acting on others. Namely, for \( X \in \mathfrak{X}(M) \) there is a linear map

\[
\text{ad} : \mathfrak{X}(M) \to \text{End} \mathfrak{X}(M) : \text{ad}_X(Y) \mapsto [X,Y]
\]

satisfying the properties for any function \( f \) on \( M \),

\[
\text{ad}_fX = f \text{ad}_X \quad \text{ad}_X(fY) = f \text{ad}_X(Y) + X(f)Y
\]

2.3. **Semisimplicity and Schur’s lemma.**

2.4. **Haar measure and Weyl’s trick.**

2.5. **Characters.**

2.6. **Examples.**

2.7. **Peter-Weyl theorem.**

3. **Maximal tori**

3.1. **Existence.**

3.2. **Weyl group.**

3.3. **Cartan subalgebras.**

4. **Classification of complex semisimple Lie algebras**

4.1. **Roots.**

4.2. **Dynkin diagrams.**

4.3. **Stieffel diagrams.**

4.4. **Classification.**

5. **Weyl character formula**

5.1. **Weights and multiplicities.**

5.2. **Examples.**
E-mail address: wentworth@jhu.edu