LIE GROUPS, LIE ALGEBRAS, AND REPRESENTATIONS
PROBLEM SET #1

(1) Let $G$ be a Lie group and $G_0$ the connected component of the identity. Show that $G_0$ is a normal subgroup of $G$.

(2) Write $U(n)$ as a finite quotient of $SU(n) \times U(1)$.

(3) Let $G(k, n)$ denote the Grassmannian of $k$-planes in $\mathbb{C}^n$. That is, each point in $G(k, n)$ corresponds to a $k$-dimensional subspace of $\mathbb{C}^n$. Realize $G(k, n)$ as a homogeneous space of compact Lie groups. (Hint: a $k$-dimensional subspace has a unitary basis)

(4) Consider the quadratic form on $\mathbb{R}^4$:

$$Q(x_1, \ldots, x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2$$

The group $SO(3, 1)$ is by definition the subgroup of $SL(4, \mathbb{R})$ that preserves $Q$:

$$SO(3, 1) = \{ A \in SL(4, \mathbb{R}) : A^T I_{3,1} A = I_{3,1} , \det A = 1 \}$$

where

$$I_{3,1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$

Find a basis for the Lie algebra $\mathfrak{so}(3, 1)$ and compute the commutators.

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