

# PRECISE CONSTANTS IN BOSONIZATION FORMULAS ON RIEMANN SURFACES I

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## 1. INTRODUCTION

The bosonization formulas on Riemann surfaces relate zeta-regularized determinants of Laplace operators acting on sections of line bundles and on scalars [AMV, VV, ABMNV, Sn, DS, F2]. They play an important role in conformal field theory and perturbative string theory (for a survey of the subject, see [DP3]). Proofs of these identities in the mathematical literature generally proceed by computing either first or second variations of certain combinations of Green's and theta functions with respect to the Riemann moduli. As a consequence, all current formulations leave undetermined constants of integration, depending only on the genus and spin, which must be evaluated by other means. There has been renewed interest recently in the precise values of these constants (cf. [DGP]).

In [W1], using ideas of Belavin-Knizhnik [BK] and D'Hoker-Phong [DP1], along with the results in [W2] on the behavior of the Arakelov metric on degenerating surfaces, values for the constants  $c_g$  associated to genus  $= g$  and spin  $= 1$  were obtained. The argument was heuristic, however, since it involved techniques from functional integration where normalizations can be somewhat arbitrary. The goal of this note is to give a rigorous proof of the result in [W1].

For the purposes of this paper, we may define  $c_g$  in terms of the relationship between Faltings' delta invariant  $\delta(M)$  of a compact Riemann surface  $M$  of genus  $g \geq 1$  and the determinant of the Laplace operator on functions with respect to the Arakelov metric (cf. [Fl]),

$$(1.1) \quad \delta(M) = c_g - 6 \log \frac{\det' \Delta_{(M, \text{Arak.})}}{\text{area}(M, \text{Arak.})}$$

The spin-1 bosonization formula states that

$$(1.2) \quad \left( \frac{\det' \Delta_{(M, \text{Arak.})}}{\text{area}(M, \text{Arak.}) \det \text{Im } \Omega} \right)^{3/4} = e^{c_g/8} \frac{\|\vartheta\| (p_1 + \cdots + p_g - z - \Delta) \prod_{i < j} G(p_i, p_j)}{\|\det \omega_i(p_j)\| \prod_{i=1}^g G(p_i, z)}$$

where  $\vartheta$  is the theta function associated to  $M$ ,  $\Delta$  is the Riemann divisor,  $\omega_i$  is a basis of abelian differentials,  $\Omega$  is the period matrix,  $G(z, w)$  is the Arakelov-Green's function, and  $\{p_i, z\} \subset M$  are generic points.

The main result is

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**Theorem 1.3.** *The value of  $c_g$  in (1.1) is*

$$\begin{aligned} c_g &= (1 - g)c_0 + gc_1 \\ c_1 &= -8 \log(2\pi) \\ c_0 &= -24\zeta'(-1) + 1 - 6 \log(2\pi) - 2 \log(2) \end{aligned}$$

where  $\zeta(s)$  is the Riemann zeta function.

**Remark 1.4.** *In [W1] one of the terms in the computation of the conformal anomaly was mistakenly neglected (see [W3] and the proof of Lemma 5.2 below). With this correction, the expression in Theorem 1.3 agrees with the result in [W1]. It is noteworthy that the path integral approach to factorization computes this constant exactly.*

**Remark 1.5.** *The constant:*

$$\begin{aligned} a_g &= (1 - g)a_0 \\ a_0 &= c_0 - c_1 = -24\zeta'(-1) + 1 + 2 \log(2\pi) - 2 \log(2) \end{aligned}$$

is called “Deligne’s constant” and is the normalization in another formulation of the bosonization formulas (see [D]). We note that this value of  $c_g$  implies that formally:  $\delta(\mathbb{P}^1) = 0$  (see [J2, §7]).

**Remark 1.6.** *In addition to [W1], expressions for  $c_g$  have previously appeared in the mathematical literature (cf. [GSo, So, J2]). These disagree slightly with the result obtained in Theorem 1.3.*

The proof of Theorem 1.3 is obtained via degeneration. Let  $M_t$  denote a family of genus  $g$  Riemann surfaces degenerating as  $t \rightarrow 0$  to a semistable nodal curve  $M_0$  with irreducible components  $M^+$  and  $M^-$  of genus  $g^+$  and  $g^-$ , respectively. We will use the following

**Theorem 1.7** ([W2], Main Theorem; see also [J1]).

$$\lim_{t \rightarrow 0} \left( \delta(M_t) + \frac{4g^+g^-}{g} \log |t| \right) = \delta(M^+) + \delta(M^-)$$

To evaluate  $c_g$  we compare this result with the degeneration of the determinant of the Laplace operator for the Arakelov metric, and then we use (1.1). The asymptotic behavior of determinants of laplacians is a widely studied subject (cf. [DP2, Sa, K, V, GIJR, H, Wo1]). The new ingredient that is useful here is the result of [BFK]. Let  $\gamma$  denote a simple closed separating curve on  $M$ , with  $R^+ \cup R^- = M - \gamma$  the connected components. Given a Riemannian metric  $h$  on  $M$ , let  $R^\pm$  have the induced metrics.

**Theorem 1.8** ([BFK], Theorem B\*).

$$\frac{\det' \Delta_{(M,h)}}{\text{area}(M,h)} = \det \Delta_{(R^+,h)} \det \Delta_{(R^-,h)} \frac{\det' \mathcal{N}_{(M,\gamma,h)}}{\ell_h(\gamma)}$$

Here, the determinants on the right-hand-side are evaluated with respect to the spectra for the Dirichlet problems on the surfaces with boundary  $R^\pm$ ,  $\ell_h(\gamma)$  denotes the length of  $\gamma$  with respect to the metric  $h$ , and  $\mathcal{N}_{(M,\gamma,h)}$  denotes the Neumann jump operator for functions on  $\gamma$ ; in this case, it is the sum of the Dirichlet-to-Neumann operators for  $R^\pm$ .

In Section 2 we show how Theorem 1.8 can be used to factorize  $\det' \Delta_M$  in terms of  $\det' \Delta_{M^\pm}$  (see Proposition 2.5). This argument is closely parallel to the one in [W1], where the sewing property of path integrals was used in lieu of Theorem 1.8. In Section 3 we make a simple observation concerning the asymptotic behavior of the Neumann jump operator  $\mathcal{N}_{(M,\gamma,h)}$  as the surface degenerates along  $\gamma$ . The key result, Proposition 3.5, is that  $\mathcal{N}_{(M,\gamma,h)}$  is equal to a universal operator on  $L^2$ -functions on the circle, modulo trace-class operators with asymptotically vanishing norm (for more detailed treatments, see [GG, PW, L, MM]). In Section 4 we briefly review the results needed on the Arakelov metric, and in Section 5 we complete the proof of Theorem 1.3.

As a final remark, we emphasize that the method described above gives the asymptotic behavior of determinants up to the zero-th order term for *any* family of conformal metrics, provided one has sufficient information on the degeneration (essentially, a  $C^0$  estimate on the metric and its curvature, with bounded growth of derivatives; see Proposition 4.6). This paper illustrates the case of the Arakelov metric, but the technique applies equally well to the hyperbolic metric, for example. Indeed, using the expansion in [Wo2] one can quickly recover the asymptotic behavior for the case of pinching along a separating curve,

$$\log \det' \Delta_{(M_t, hyp.)} = (1/6) \log |t| + O(1)$$

that is a consequence of the (more precise) expression given in [Wo1, Theorem 5.3], without passing through the explicit evaluation of determinants in terms of Selberg zeta functions. This also explains why evaluating the next order term is more difficult in this case, since the procedure in [Wo2] relates the hyperbolic metric on  $M_t$  to the complete hyperbolic metrics on the punctured surfaces  $M^\pm - \{p^\pm\}$ , and not to the hyperbolic metrics on the closed surfaces  $M^\pm$ . The conformal factors relating these two are, of course, quite complicated (cf. [JL]).

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## 2. FACTORIZATION OF DETERMINANTS

Let  $M^+$  and  $M^-$  be a pair of closed Riemann surfaces with genera  $g^\pm \geq 1$ . For a complex parameter  $t$ ,  $0 < |t| \leq 1$  we construct a degenerating family  $M_t$  of closed surfaces of genus  $g = g^+ + g^-$  using the ‘‘plumbing construction’’:  $z^+ z^- = t$ , for local coordinates  $z^\pm$  on  $M^\pm$  centered at points  $p^\pm$  (for more details, see [W2]). Let  $\gamma_t$  denote the curve in  $M_t$  given by  $|z^+| = |z^-| = |t|^{1/2}$ . If we set  $B_t = \{z \in \mathbb{C} : |z| \leq |t|^{1/2}\}$ , identify  $z$  with  $z^\pm$ , and let  $R_t^\pm = M^\pm - B_t$ , then  $M_t - \gamma_t$  is conformally equivalent to the disjoint union  $R_t^+ \cup R_t^-$ . Fix conformal metrics  $h^\pm$  on  $M^\pm$ . For

simplicity, we will assume in advance that  $z^\pm$  are normalized in the sense that with respect to these coordinates  $h^\pm(0) = 1$ . Let  $h_t$ ,  $t \neq 0$  be a family of conformal metrics on  $M_t$ . Then on  $R_t^\pm$  there are conformal factors

$$(2.1) \quad h_t|_{M_t - \gamma_t} = h^\pm e^{2\sigma_t^\pm}$$

Set

$$(2.2) \quad \mathcal{Q}_{(M_t, h_t)} = \frac{\det' \Delta_{(M_t, h_t)}}{\text{area}(M_t, h_t)}$$

where  $\det' \Delta$  denotes the zeta-regularized determinant of the (positive) Laplace-Beltrami operator over the nonzero spectrum  $\{\lambda_j\}_{j=1}^\infty$ :

$$\log \det' \Delta = -\zeta'_\Delta(0), \quad \zeta_\Delta(s) = \sum_{j=1}^\infty \lambda_j^{-s}$$

Similarly, we define

$$\mathcal{Q}_{(M^\pm, h^\pm)} = \frac{\det' \Delta_{(M^\pm, h^\pm)}}{\text{area}(M^\pm, h^\pm)}$$

We wish to derive an expression for  $\mathcal{Q}_{M_t}$  in terms of  $\mathcal{Q}_{M^\pm}$ . To do this, we pass through the relative determinants. Let

$$\mathcal{Q}_{(R_t^\pm, h_t)} = \det \Delta_{(R_t^\pm, h_t)}, \quad \mathcal{Q}_{(R_t^\pm, h^\pm)} = \det \Delta_{(R_t^\pm, h^\pm)}$$

denote the determinants for the Laplacian on  $R_t^\pm$  with respect to the restriction of the metric  $h_t$  on the one hand, and the metrics  $h^\pm$  on the other. In this note we always choose Dirichlet boundary conditions on a manifold with boundary. Since  $h_t$  and  $h^\pm$  are conformally related as in (2.1), we have the Polyakov-Alvarez formula (cf. [A]):

$$(2.3) \quad \mathcal{Q}_{(R_t^\pm, h_t)} = \mathcal{Q}_{(R_t^\pm, h^\pm)} e^{S_{(R_t^\pm, h^\pm)}(\sigma_t^\pm)}$$

$$(2.4) \quad S_{(R, h)}(\sigma) = -\frac{1}{12\pi} \int_R dA_h \{2K_h \sigma + |\nabla \sigma|^2\} - \frac{1}{12\pi} \int_{\partial R} ds_h \{2k_h \sigma + 3\partial_n \sigma\}$$

where  $K_h$  and  $k_h$  are the Gauss and geodesic curvatures, respectively, for the metric  $h$ . Let  $\mathcal{N}_t$  denote the Neumann jump operator for  $(M_t, \gamma_t, h_t)$ , and  $\mathcal{N}_t^\pm$  the Neumann jump operators for the surfaces  $(M^\pm, \gamma_t^\pm, h^\pm)$ , where  $\gamma_t^\pm = \partial R_t^\pm$ . Finally, we set

$$\mathcal{Q}_{B_t} = \det \Delta_{B_t}, \quad \mathcal{Q}_{(B_t, h)} = \det \Delta_{(B_t, h)}$$

where  $B_t$  has the Euclidean metric, or a general metric  $h$ , respectively. Then we have the following

**Proposition 2.5.**

$$\begin{aligned} \lim_{t \rightarrow 0} \left\{ \mathcal{Q}_{(M_t, h_t)} (\mathcal{Q}_{B_t})^2 \exp \left\{ -S_{(R_t^+, h^+)}(\sigma_t^+) - S_{(R_t^-, h^-)}(\sigma_t^-) \right\} \frac{\ell_{h_t}(\gamma_t) \det' \mathcal{N}_t^+ \det' \mathcal{N}_t^-}{\ell_{h^+}(\gamma_t) \ell_{h^-}(\gamma_t) \det' \mathcal{N}_t} \right\} \\ = \mathcal{Q}_{(M^+, h^+)} \mathcal{Q}_{(M^-, h^-)} \end{aligned}$$

*Proof.* Applying Theorem 1.8 and (2.3) we have

$$\mathcal{Q}_{(M_t, h_t)} = \mathcal{Q}_{(R_t^+, h^+)} \mathcal{Q}_{(R_t^-, h^-)} \exp\{S_{(R_t^+, h^+)}(\sigma_t^+) + S_{(R_t^-, h^-)}(\sigma_t^-)\} \frac{\det' \mathcal{N}_t}{\ell_{h_t}(\gamma_t)}$$

Using Theorem 1.8 again for  $(M^\pm, h^\pm)$ :

$$\mathcal{Q}_{(M^\pm, h^\pm)} = \mathcal{Q}_{(R_t^\pm, h^\pm)} \mathcal{Q}_{(B_t, h^\pm)} \frac{\det' \mathcal{N}_t^\pm}{\ell_{h^\pm}(\gamma_t^\pm)}$$

Now  $\mathcal{Q}_{(B_t, h^\pm)}$  and  $\mathcal{Q}_{B_t}$  are related by the Liouville action (2.4) for a smooth conformal factor. Hence,  $\mathcal{Q}_{(B_t, h^\pm)}/\mathcal{Q}_{B_t} \rightarrow 1$  as  $t \rightarrow 0$  (recall the normalization  $h^\pm(0) = 1$ ). The result follows.  $\square$

### 3. ASYMPTOTICS OF THE NEUMANN JUMP OPERATOR

Let  $M$  be a compact Riemann surface and  $z = re^{i\theta}$  a local coordinate. For a real parameter  $\varepsilon$ ,  $0 < \varepsilon \leq 1$ , set  $B(\varepsilon) = \{x \in M : |z(x)| < \varepsilon\}$ , and  $R(\varepsilon) = M - B(\varepsilon)$  (n.b. in the notation of the previous section,  $B(\varepsilon) = B_t$  and  $R^\pm(\varepsilon) = R_t^\pm$ , where  $|t| = \varepsilon^2$ ). Assume  $M$  is equipped with a conformal metric that is euclidean  $|dz|^2$  in a neighborhood of  $B(1)$ . Let  $\mathcal{P}_{R(\varepsilon)} : L^2(S^1) \rightarrow L^2(S^1)$  (resp.  $\mathcal{P}_{B(\varepsilon)}$ ) denote the Dirichlet-to-Neumann operator for  $R(\varepsilon)$  (resp.  $B(\varepsilon)$ ). Normals are always taken to be outward pointing. Note that  $\ker \mathcal{P}_{R(\varepsilon)}$  consists of the constants, and by Stokes' theorem,  $\mathcal{P}_{R(\varepsilon)} : L_0^2(S^1) \rightarrow L_0^2(S^1)$ , where  $L_0^2(S^1)$  is the subspace of  $L^2(S^1)$  orthogonal to the constants. Define the following auxiliary operators on  $L^2(S^1)$ :

$$\begin{aligned} \mathcal{T}_\varepsilon &: \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \mapsto \sum_{n \in \mathbb{Z}} \left( \frac{\varepsilon^n - \varepsilon^{-n}}{\varepsilon^n + \varepsilon^{-n}} \right) a_n e^{in\theta} \\ \mathcal{U}_\varepsilon^\pm &: \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \mapsto \frac{1}{2} \sum_{n \in \mathbb{Z}} (\varepsilon^n \pm \varepsilon^{-n}) a_n e^{in\theta} \\ \mathcal{V} &: \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \mapsto \sum_{n \in \mathbb{Z}} n a_n e^{in\theta} \\ |\mathcal{V}| &: \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \mapsto \sum_{n \in \mathbb{Z}} |n| a_n e^{in\theta} \end{aligned}$$

**Remark 3.1.** On  $L_0^2(S^1)$ , the operators  $\mathcal{V}(\mathcal{U}_\varepsilon^-)^{-1}$  and  $\{\mathcal{V}(\mathcal{T}_\varepsilon)^{-1} + |\mathcal{V}|\}$  are trace-class with norm tending to zero as  $\varepsilon \rightarrow 0$ .

**Remark 3.2.** By direct computation:  $\varepsilon \mathcal{P}_{B(\varepsilon)} = |\mathcal{V}|$ .

We also define the (unbounded) operator  $\mathcal{E}_{R(\varepsilon)} : L^2(S^1) \rightarrow L^2(S^1)$  as follows. For  $f \in L^2(S^1)$ , let  $u$  be the harmonic function on  $R(1)$  with boundary values  $f$ . Extend  $u$  to a harmonic function on  $R(\varepsilon)$  (also denoted  $u$ ). Then  $\mathcal{E}_{R(\varepsilon)}(f) = u|_{\partial R(\varepsilon)}$ . We have the following simple

**Lemma 3.3.**  $\|\mathcal{E}_{R(\varepsilon)}(f)\| \geq \|f\|$  for all  $0 < \varepsilon \leq 1$  and all  $f \in L^2(S^1)$ . In particular,  $\mathcal{E}_{R(\varepsilon)}^{-1}$  is uniformly bounded as  $\varepsilon \rightarrow 0$ .

*Proof.* It suffices to prove the estimate for  $f$  smooth. Let  $u$  be harmonic on  $R(\varepsilon)$  with  $u|_{\partial R(1)} = f$ . Then

$$\begin{aligned}
\|f\|^2 - \|\mathcal{E}_{R(\varepsilon)}(f)\|^2 &= \int_0^{2\pi} d\theta \{u^2(1, \theta) - u^2(\varepsilon, \theta)\} \\
&= \int_\varepsilon^1 dr \frac{d}{dr} \int_0^{2\pi} d\theta u^2(r, \theta) \\
&= 2 \int_\varepsilon^1 \frac{dr}{r} \int_0^{2\pi} r d\theta u(r, \theta) \frac{\partial u}{\partial r}(r, \theta) \\
&= -2 \int_\varepsilon^1 \frac{dr}{r} \int_{\partial R(r)} ds u \partial_n u \\
&= -2 \int_\varepsilon^1 \frac{dr}{r} \int_{R(r)} dA |\nabla u|^2 \\
&\leq 0
\end{aligned}$$

□

We also have

**Lemma 3.4.**  $\mathcal{E}_{R(\varepsilon)}$  preserves the orthogonal splitting  $L^2(S^1) = \mathbb{C} \oplus L_0^2(S^1)$ .

*Proof.* Let  $u$  be a harmonic function on  $R(\varepsilon)$  with  $u|_{\partial R(1)} = f$ ,  $u|_{\partial R(\varepsilon)} = g$ . Then applying Green's theorem with the harmonic function  $v = \log r$  on the annulus  $B(1) - B(\varepsilon)$ ,

$$\begin{aligned}
0 &= \int_{\partial(R(\varepsilon)-R(1))} (u \partial_n v - v \partial_n u) ds \\
&= \int_{S^1} f d\theta - \int_{S^1} g d\theta - \int_{\partial R(\varepsilon)} \mathcal{P}_{R(\varepsilon)}(g) \log \varepsilon ds
\end{aligned}$$

Since  $\mathcal{P}_{R(\varepsilon)}(g) \in L_0^2(S^1)$ , we have

$$\int_{S^1} f d\theta = \int_{S^1} g d\theta = \int_{S^1} \mathcal{E}_{R(\varepsilon)}(f) d\theta$$

□

The following result shows that the operators  $\varepsilon \mathcal{P}_{R(\varepsilon)}$  and  $\varepsilon \mathcal{P}_{B(\varepsilon)}$  are asymptotically close as  $\varepsilon \rightarrow 0$ .

**Proposition 3.5.** On  $L_0^2(S^1)$ ,  $\varepsilon \mathcal{P}_{R(\varepsilon)} = -\mathcal{V}(\mathcal{T}_\varepsilon)^{-1} + \mathcal{V}(\mathcal{U}_\varepsilon^-)^{-1} \mathcal{E}_{R(\varepsilon)}^{-1}$ .

*Proof.* First, note that by Lemma 3.4 the composition  $(\mathcal{U}_\varepsilon^-)^{-1} \mathcal{E}_{R(\varepsilon)}^{-1}$  is well-defined on  $L_0^2(S^1)$ . With this understood, the result follows by a direct calculation. Namely, by expanding functions in Fourier modes, one computes

$$\begin{aligned}
\mathcal{E}_{R(\varepsilon)} &= \mathcal{U}_\varepsilon^+ - \mathcal{V}^{-1} \mathcal{U}_\varepsilon^- \mathcal{P}_{R(1)} \\
-\varepsilon \mathcal{P}_{R(\varepsilon)} \mathcal{E}_{R(\varepsilon)} &= \mathcal{V} \mathcal{U}_\varepsilon^- - \mathcal{U}_\varepsilon^+ \mathcal{P}_{R(1)}
\end{aligned}$$

The proposition is then a consequence of these two facts.  $\square$

**Corollary 3.6.** *Let  $\Lambda_t$  be any of  $\det' \mathcal{N}_t / \ell_{h_t}(\gamma_t)$ , or  $\det' \mathcal{N}_t^\pm / \ell_{h^\pm}(\gamma_t^\pm)$  from Section 2. Then*

$$\lim_{t \rightarrow 0} \Lambda_t = 1/2$$

*Proof.* Consider  $\mathcal{N}_t^+$ , the argument being similar for the other two cases. By conformal invariance of  $\det' \mathcal{N}_t^+ / \ell_{h^+}(\gamma_t^+)$  (cf. [GG], the method in [EW], or note that this follows from (2.3) and Theorem 1.8) it suffices to consider the locally euclidean case. Also, notice that

$$(3.7) \quad \zeta_{\mathcal{N}_t^+}(0) = -1$$

for all  $t$ . Indeed, if we scale the metric by  $c^2$ ,

$$\ell_{h^+}(\gamma_t^+) \mapsto c \ell_{h^+}(\gamma_t^+), \quad \mathcal{N}_t^+ \mapsto c^{-1} \mathcal{N}_t^+$$

Then by conformal invariance

$$\log \frac{\det \mathcal{N}_t^+}{\ell_{h^+}(\gamma_t^+)} = \log \frac{\det(c^{-1} \mathcal{N}_t^+)}{c \ell_{h^+}(\gamma_t^+)} = -(\zeta_{\mathcal{N}_t^+}(0) + 1) \log c + \log \frac{\det \mathcal{N}_t^+}{\ell_{h^+}(\gamma_t^+)}$$

and since  $c$  is arbitrary, (3.7) holds. As a consequence,

$$(3.8) \quad \begin{aligned} \log \det(|t|^{1/2} \mathcal{N}_t^+) &= \zeta_{\mathcal{N}_t^+}(0) \log |t|^{1/2} + \log \det \mathcal{N}_t^+ \\ &= -\log |t|^{1/2} + \log \det \mathcal{N}_t^+ \\ \log \det(|t|^{1/2} \mathcal{N}_t^+) &= \log \frac{\det \mathcal{N}_t^+}{\ell_{h^+}(\gamma_t^+)} + \log(2\pi) \end{aligned}$$

On the other hand, by Proposition 3.5, Remarks 3.1 and 3.2, and Lemma 3.3, it follows that

$$|t|^{1/2} \mathcal{N}_t^+ = 2|\mathcal{V}| + \{\text{trace-class}\}$$

where the trace norm of the remainder tends to zero as  $t \rightarrow 0$ . Hence by (3.8) and [L, Lemma 4.1],

$$\lim_{t \rightarrow 0} \log \Lambda_t = \lim_{t \rightarrow 0} \left( \log \det(|t|^{1/2} \mathcal{N}_t^+) - \log(2\pi) \right) = \log \det(2|\mathcal{V}|) - \log(2\pi)$$

Now  $\text{spec}(2|\mathcal{V}|) = \{2n\}_{n=0}^\infty$ , and each of the nonzero eigenvalues has multiplicity 2. Hence,  $\zeta_{2|\mathcal{V}|}(s) = 2^{1-s} \zeta(s)$ . The result then follows from the special values of the Riemann zeta function:  $\zeta(0) = -1/2$ ,  $\zeta'(0) = -(1/2) \log(2\pi)$ .  $\square$

#### 4. THE ARAKELOV METRIC

Recall the definition of the Arakelov metric (cf. [Ar, Fl, J1, W2, F2]). Given a compact Riemann surface  $M$  of genus  $g \geq 1$ , let  $\{A_i, B_i\}_{i=1}^g$  be a symplectic set of generators of  $H_1(M)$  and choose  $\{\omega_i\}_{i=1}^g$  to be a basis of abelian differentials normalized such that  $\int_{A_i} \omega_j = \delta_{ij}$ . Let  $\Omega_{ij} = \int_{B_i} \omega_j$  be the associated period matrix with theta function  $\vartheta$ . Set

$$(4.1) \quad \mu = \frac{\sqrt{-1}}{2g} \sum_{i,j=1}^g (\Im \Omega)_{ij}^{-1} \omega_i \wedge \bar{\omega}_j$$

Then  $\int_M \mu = 1$ . The Arakelov-Green's function  $G(z, w)$  is symmetric with a zero of order one along the diagonal satisfying

$$(4.2) \quad \partial_z \partial_{\bar{z}} \log G(z, w) = \pi \sqrt{-1} \mu_{z\bar{z}}, \quad z \neq w;$$

$$(4.3) \quad \int_M \mu(z) \log G(z, w) = 0$$

and the Arakelov metric  $h = h_{z\bar{z}} |dz|^2$  is defined by

$$(4.4) \quad \log h_{z\bar{z}} = 2 \lim_{w \rightarrow z} \{ \log G(z, w) - \log |z - w| \}$$

The Arakelov metric is “admissible” in the sense of [Fl]; hence,

$$(4.5) \quad \text{Ric}(h) = 4\pi \sqrt{-1} (g - 1) \mu$$

For more details we refer to the papers cited above.

Consider now the situation in Section 2 where  $h_t$  and  $h^\pm$  denote the Arakelov metrics on  $M_t$  and  $M^\pm$ , respectively (see [Ar, Fl]). The purpose of this section is to prove

**Proposition 4.6** (cf. [W2], eq. (8.1)). *Let  $h_t = e^{2\sigma_t^\pm} h^\pm$ . Then for  $z \in R_t^\pm$ ,*

$$(4.7) \quad \sigma_t^\pm(z) = \left( \frac{g^\mp}{g} \right)^2 \log |t| - 2 \left( \frac{g^\mp}{g} \right) \log G^\pm(z, p^\pm) + r^\pm(t, z)$$

where  $\lim_{t \rightarrow 0} \sup_{z \in R_t^\pm} |r^\pm(t, z)| = 0$ . Moreover, if  $z^\pm$  are the plumbing coordinates then there is a constant  $C > 0$  independent of  $t$  such that

$$(4.8) \quad \sup_{|z^\pm| = |t|^{1/2}} |\partial_{z^\pm} \sigma_t^\pm(z^\pm)| \leq C |t|^{-1/2}$$

In the statement above,  $G^\pm(z, w)$  denote the Arakelov-Green's functions on  $M^\pm$ . The uniformity of the estimate is an improvement on the result of [W2] and is made possible by the explicit expression for the Arakelov metric in [F2] (eq. (4.28) below). We require two preliminary technical results.

**Lemma 4.9.** *Let  $\omega_t$  be a holomorphic family of abelian differentials on  $M_t$  such that (pointwise)  $\omega_t(z) \rightarrow \omega_0(z)$  for  $z \in M^+ - \{p^+\}$ , and  $\omega_t(z) \rightarrow 0$  for  $z \in M^- - \{p^-\}$ , where  $\omega_0$  is an abelian differential on  $M^+$ . Let  $W_t^\pm, W_0^\pm$  be the local expressions for  $\omega_t$  and  $\omega_0$  in the plumbing coordinates  $z^\pm$ . Then there are functions  $f(t, x), g(t, x)$ , analytic in a neighborhood of  $(0, 0)$ , and  $h(t)$  analytic in a neighborhood of 0, such that*

$$(4.10) \quad W_t^+(z^+) = \frac{t}{z^+} h(t) + f(t, z^+ + t/z^+) + \frac{t^2}{(z^+)^2} g(t, z^+ + t/z^+)$$

$$(4.11) \quad W_t^-(z^-) = -\frac{t}{z^-} h(t) - \frac{t}{(z^-)^2} f(t, z^- + t/z^-) - t g(t, z^- + t/z^-)$$

In particular, if we assume  $f(0, 0) \neq 0$ , then for  $|z^\pm|$  sufficiently small,

$$(4.12) \quad \log |W_t^+(z^+)| = \log |W_0^+(z^+)| + r^+(t, z^+)$$

$$(4.13) \quad \log |W_t^-(z^-)| = \log |t| + \log |\dot{W}_0^-(z^-)| + r^-(t, z^-)$$

where

$$\begin{aligned} W_0^+(z^+) &= f(0, z^+) \\ \dot{W}_0^-(z^-) &= -\frac{h(0)}{z^-} - \frac{f(0, z^-)}{(z^-)^2} - g(0, z^-) \end{aligned}$$

and  $\lim_{t \rightarrow 0} \sup_{|z^\pm| \geq |t|^{1/2}} |r^\pm(t, z^\pm)| = 0$ . Moreover, there is a constant  $C$  independent of  $t$  such that

$$(4.14) \quad \sup_{|z^\pm| = |t|^{1/2}} |\partial_{z^\pm} W_t^\pm(z^\pm)| \leq C|t|^{-1/2}$$

*Proof.* By [F1, p. 40] we have an expansion

$$(4.15) \quad \omega_t = \sum_{m \geq 0} a_m(t) \mathcal{X}^m d\mathcal{X} + \sum_{n \geq 0} b_n(t) \mathcal{X}^n \frac{d\mathcal{X}}{\mathcal{Y}}$$

where

$$(4.16) \quad \mathcal{X} = (1/2)(z^+ + z^-) = (1/2)(z^+ + t/z^+)$$

$$(4.17) \quad \mathcal{Y} = (1/2)(z^+ - z^-) = -(1/2)(z^- - t/z^-)$$

The  $a_m(t)$ ,  $b_n(t)$  are analytic near  $t = 0$ , and the series are convergent for  $(\mathcal{X}, t)$  in a neighborhood of  $(0, 0)$ . Moreover, using the assumption that  $\omega_t(z) \rightarrow 0$  on  $M^- - \{p^-\}$ , we have  $b_0(0) = 0$ , and  $b_{n+1}(0) = a_n(0)$  for all  $m \geq 0$ . Now, substituting the expressions for  $\mathcal{X}$ ,  $d\mathcal{X}$ , and  $\mathcal{Y}$ , we find

$$\begin{aligned} W_t^+(z^+) &= \sum_{m \geq 0} \frac{a_m(t)}{2^{m+1}} \left(1 - \frac{t}{(z^+)^2}\right) \left(z^+ + \frac{t}{z^+}\right)^m + \sum_{n \geq 0} \frac{b_n(t)}{2^n} \frac{1}{z^+} \left(z^+ + \frac{t}{z^+}\right)^n \\ &= \frac{b_0(t)}{t} \frac{t}{z^+} + \sum_{m \geq 0} \frac{1}{2^{m+1}} (a_m(t) + b_{m+1}(t)) \left(z^+ + \frac{t}{z^+}\right)^m \\ &\quad - \sum_{n \geq 0} \frac{1}{2^{n+1}} \frac{(a_n(t) - b_{n+1}(t))}{t} \frac{t^2}{(z^+)^2} \left(z^+ + \frac{t}{z^+}\right)^n \end{aligned}$$

Since  $b_0(t)/t$  and  $(a_n(t) - b_{n+1}(t))/t$  are regular at  $t = 0$ , this gives the expression in (4.10) with

$$\begin{aligned} h(t) &= \frac{b_0(t)}{t} \\ f(t, x) &= \sum_{m \geq 0} \frac{1}{2^{m+1}} (a_m(t) + b_{m+1}(t)) x^m \\ g(t, x) &= - \sum_{n \geq 0} \frac{1}{2^{n+1}} \frac{(a_n(t) - b_{n+1}(t))}{t} x^n \end{aligned}$$

(4.11) follows from (4.10) and the fact that  $dz^+ = -tdz^-(z^-)^2$ .

For fixed  $z^+$  write

$$\begin{aligned} F(t) &= f(t, z^+ + t/z^+) + \frac{t}{z^+}h(t) + \frac{t^2}{(z^+)^2}g(t, z^+ + t/z^+) \\ F'(t) &= \partial_1 f + \frac{1}{z^+}\partial_2 f + \frac{1}{z^+}h + \frac{t}{z^+}h' + \frac{2t}{(z^+)^2}g \\ &\quad + \frac{t^2}{(z^+)^2}\partial_1 g(t, z^+ + t/z^+) + \frac{t^2}{(z^+)^3}\partial_2 g(t, z^+ + t/z^+) \end{aligned}$$

Hence, for  $|z^+| \geq |t|^{1/2}$ , we have  $F(t) = f(0, 0) + O(|t|^{1/2})$ , uniformly. This implies (4.12). For fixed  $z^-$  write

$$\begin{aligned} G(t) &= f(t, z^- + t/z^-) + z^-h(t) + (z^-)^2g(t, z^- + t/z^-) \\ G'(t) &= \partial_1 f + \frac{1}{z^-}\partial_2 f + z^-h' + (z^-)^2\partial_1 g(t, z^- + t/z^-) + z^-\partial_2 g(t, z^- + t/z^-) \end{aligned}$$

It follows that  $G(t) = G(0) + O(|t|^{1/2})$ , uniformly for  $|z^-| \geq |t|^{1/2}$ . By assumption,  $G(0) \neq 0$  for  $|z^-|$  sufficiently small, and hence

$$W_t^-(z^-) = -\frac{t}{(z^-)^2}G(0)(1 + O(|t|^{1/2}))$$

so the estimate (4.13) follows. The estimate (4.14) follows immediately from (4.10) and (4.11).  $\square$

For the second result, let  $E_t(z, w)$ ,  $E^\pm(z, w)$  denote the Schottky prime forms on  $M_t$ ,  $M^\pm$  (cf. [F1, F2] and eq. (4.22) below). When evaluated at points on the Riemann surface, we will assume that lifts to a fixed fundamental domain have been chosen. Also, when points are in the pinching region we will assume the expressions for the prime forms are given with respect to the coordinates  $z^\pm$ . The following asymptotics are well-known. The point again is the uniformity of the estimates.

**Lemma 4.18.** *For  $z, w \in R_t^+$ ,*

$$(4.19) \quad \log |E_t(z, w)| = \log |E^+(z, w)| + r_1^+(z, w, t)$$

where for fixed  $w$ ,  $\lim_{t \rightarrow 0} \sup_{z \in R_t^+} |r_1^+(z, w, t)| = 0$  (and similarly for fixed  $z$ ). A similar result holds for points in  $R_t^-$ . For  $z \in R_t^+$  and  $w \in R_t^-$ ,

$$(4.20) \quad \log |E_t(z, w)| = -\log |t|^{1/2} + \log |E^+(z, p^+)| + \log |E^-(w, p^-)| + r_2(z, w, t)$$

where for fixed  $w$ ,  $\lim_{t \rightarrow 0} \sup_{z \in R_t^+} |r_2(z, w, t)| = 0$  (and similarly for fixed  $z$ ). Moreover, if  $z^\pm$  are the plumbing coordinates and  $w$  is fixed, then there is a constant  $C > 0$  independent of  $t$  such that

$$(4.21) \quad \sup_{|z^\pm| = |t|^{1/2}} |\partial_{z^\pm} E_t^\pm(z^\pm, w)| \leq C|t|^{-1/2}$$

*Proof.* Recall that the prime form may be defined as follows. Let  $\vartheta(Z)$ ,  $Z \in \mathbb{C}^g$ , denote the theta function for the period matrix  $\Omega$ . Choose a nonsingular  $\delta \in \Theta$ , where  $\Theta \subset \text{Jac}(M)$  is the theta

divisor in the Jacobian of  $M$ . Then by [F1, Corollary 2.3],

$$(4.22) \quad -E^2(z, w) = \frac{\vartheta(z - w + \delta)\vartheta(z - w - \delta)}{H_\delta(z)H_\delta(w)}$$

where

$$(4.23) \quad H_\delta(z) = \sum_{i=1}^g \partial_{Z_i} \vartheta(\delta) \omega_i(z)$$

Now consider the degenerating family  $M_t$ . The collection  $\{A_i, B_i\}_{i=1}^g$  may be chosen so that  $\{A_i, B_i\}_{i=1}^{g^+}$  is a symplectic homology basis for  $M^+$  and  $\{A_i, B_i\}_{i=g^++1}^g$  is a symplectic homology basis for  $M^-$ . Let  $\omega_{i,t}, \omega_i^\pm$  be abelian differentials on  $M_t$  and  $M^\pm$ , respectively, normalized with respect to  $\{A_i\}$ , and let  $\Omega_t$  and  $\Omega^\pm$  be the associated period matrices. Then as  $t \rightarrow 0$ ,

$$(4.24) \quad \omega_{i,t}(z) \longrightarrow \begin{cases} \omega_i^+(z) & i \leq g^+, z \in M^+ - \{p^+\} \\ 0 & i \leq g^+, z \in M^- - \{p^-\} \\ \omega_i^-(z) & i > g^+, z \in M^- - \{p^-\} \\ 0 & i > g^+, z \in M^+ - \{p^+\} \end{cases}$$

and  $\Omega_t$  becomes block diagonal  $(\Omega^+, \Omega^-)$  (cf. [F1, p. 38]). Choose  $\delta_t$  so that

$$\lim_{t \rightarrow 0} \delta_t = (\delta^+, \delta^-) \in \Theta^+ \times \text{Jac}(M^-)$$

If  $z, w \in R_t^+$ , we also have  $\vartheta_t(z - w + \delta_t) \rightarrow \vartheta^+(z - w + \delta^+) \vartheta^-(\delta^-)$ , and this is uniform in  $z, w$  (cf. [W2, Sect. 3 and Prop. A.1]). For simplicity and without loss of generality, assume  $\vartheta^-(\delta^-) \neq 0$  and  $H_{\delta^+}^+(p^+) \neq 0$ . Then from (4.23),

$$(4.25) \quad H_{\delta_t}(z) \rightarrow H_{\delta^+}^+(z) \vartheta^-(\delta^-)$$

and by Lemma 4.9 this is uniform for  $z \in R_t^+$ . Hence, the expression (4.22) immediately implies (4.19). Now suppose  $z, w \in R_t^-$ . From [W2, Prop. 3.8] we have

$$(4.26) \quad \vartheta_t(z - w \pm \delta_t) = t \frac{\vartheta^-(z - p^- \pm \delta^-) \vartheta^-(w - p^- \mp \delta^-) E^-(z, w)}{\vartheta^-(\delta^-) E^-(z, p^-) E^-(w, p^-)} H_{\delta^+}^+(p^+) + O(|t|^2)$$

where the  $O(|t|^2)$  term depends on two derivatives of  $\vartheta(Z, \Omega)$  with respect to  $\Omega_{ij}$  and is smooth in  $Z$ . Hence, again this is uniform. We then also have,

$$(4.27) \quad H_{\delta_t}(z) = t \frac{\vartheta^-(z - p^- + \delta^-) \vartheta^-(z - p^- - \delta^-)}{\vartheta^-(\delta^-) (E^-(z, p^-))^2} H_{\delta^+}^+(p^+) + O(|t|^2)$$

Again, (4.19) follows. The uniformity for the remainder term in  $\log |H_{\delta_t}(z)|$  comes from (4.13) as applied to  $H_{\delta_t}(z)$  and  $H_{\delta_t}(w)$ , and as mentioned, the expansion (4.26) is also uniform. The expression (4.20) follows. For (4.21), note that from (4.22),

$$2\partial_z E_t(z, w) = E_t(z, w) \left\{ \frac{\sum_{i=1}^g \partial_{Z_i} \vartheta_t(z - w + \delta_t) \omega_i(z)}{\vartheta_t(z - w + \delta_t)} + \frac{\sum_{i=1}^g \partial_{Z_i} \vartheta_t(z - w - \delta_t) \omega_i(z)}{\vartheta_t(z - w - \delta_t)} - \frac{\partial_z H_{\delta_t}(z)}{H_{\delta_t}(z)} \right\}$$

Now fix  $w \in R_t^-$ . By (4.19),  $|E_t(z, w)|$  is uniformly bounded from above for  $|z^+| = |t|^{1/2}$ , and  $|H_{\delta_t}(z^+)|$  is uniformly bounded from below (see Lemma 4.9). Similarly,  $|\vartheta_t(z - w \pm \delta_t)| \sim |t|^{1/2}$  (see (4.26)). Hence,

$$\sup_{|z^\pm|=|t|^{1/2}} |\partial_{z^+} E_t(z^+, w)| \leq C_1 |t|^{-1/2} + C_2 \sup_{|z^\pm|=|t|^{1/2}} |\partial_{z^+} H_{\delta_t}(z^+)|$$

and the desired bound in (4.21) follows from (4.14) applied to  $H_{\delta_t}(z)$ . The case  $w \in R_t^+$  is similar. Finally, in case  $z \in R_t^+$ ,  $w \in R_t^-$ , use  $\vartheta_t(z - w + \delta_t) \rightarrow \vartheta^+(z - p^+ + \delta^+) \vartheta^-(w - p^- + \delta^-)$ , (4.25), and (4.27) to obtain (4.20). The uniformity and statement about derivatives follows as above. This completes the proof.  $\square$

*Proof of Proposition 4.6.* From [F2, eq. (1.31)], the Arakelov metric may be expressed in local conformal coordinates as

$$(4.28) \quad h(z) = C(M) |s(z)|^{4/g} \exp\{(4/g(g-1))B[k^z, k^z]\} |dz|^2$$

where  $C(M)$  is an explicit constant depending on  $M$  but independent of  $z$ ,  $k^z$  is the vector of Riemann constants associated to  $z$ ,

$$k_j^z = \frac{1 + \Omega_{jj}}{2} + \sum_{i \neq j} \int_{A_i} \omega_i(x) \int_x^z \omega_j$$

and

$$B[k^z, k^z] = \pi \sum_{i=1}^g (\Im \Omega)_{ij}^{-1} (\Im k^z)_i (\Im k^z)_j$$

The function  $s(z)$  is given in [F2, Prop. 1.2]. Important here is the relation

$$(4.29) \quad \frac{s(z)}{s(z_0)} = \frac{\vartheta(\sum_{i=1}^g p_i - z - \Delta)}{\vartheta(\sum_{i=1}^g p_i - z_0 - \Delta)} \prod_{i=1}^g \frac{E(p_i, z)}{E(p_i, z_0)}$$

where  $p_1, \dots, p_g$  are generic points. Using Theorem 1.7 one can in fact recover the asymptotics in [W2, eq. (8.1)] from (4.28). Relevant to the proof of Proposition 4.6, however, is the uniformity. From (4.28) and (4.29) this amounts to uniformity and bound on derivatives of the prime form, since the uniformity of the theta function and Riemann constants follows from the results in [W2]. Hence, Proposition 4.6 is a consequence of Lemma 4.18.  $\square$

## 5. PROOF OF THE MAIN THEOREM

We now apply the formulation of Sections 2 and 3 to  $\mathcal{Q}_{(M_t, h_t)}$  and  $\mathcal{Q}_{(M^\pm, h^\pm)}$ , where  $h_t$  and  $h^\pm$  are the Arakelov metrics on  $M_t$  and  $M^\pm$ , respectively. The main result is

**Proposition 5.1.**

$$\lim_{t \rightarrow 0} \left( \log \mathcal{Q}_{(M_t, h_t)} - \frac{2g^+ g^-}{3g} \log |t| \right) = \log(\mathcal{Q}_{(M^+, h^+)} \mathcal{Q}_{(M^-, h^-)}) - \frac{c_0}{6}$$

where  $c_0$  is defined in Theorem 1.3.

For this we require

**Lemma 5.2.** *In the notation of Section 2,*

$$\lim_{t \rightarrow 0} \left\{ S_{(R_t^+, h^+)}(\sigma_t^+) + S_{(R_t^-, h^-)}(\sigma_t^-) + \left( \frac{1}{3} - \frac{2g^+g^-}{3g} \right) \log |t| + 1 \right\} = 0$$

*Proof.* This is essentially [W1, eq. (3.5)]. However, there is an error in the computation due to the omission of the last term in (2.4). See [W3] or (5.8) below for the correction. Moreover, the uniformity of the asymptotics of the Arakelov metric, established in the previous section, was assumed in that paper. Hence, for the sake of completeness, we include the full computation here. Let  $\mu_t, \mu^\pm$  be the forms corresponding to (4.1) on  $M_t$  and  $M^\pm$ . It follows from (4.24) that

$$(5.3) \quad \mu_t \longrightarrow (g^\pm/g)\mu^\pm$$

and from Lemma 4.9,  $\mu_t$  is uniformly bounded in the plumbing coordinates as  $t \rightarrow 0$ . By (4.5) we have  $K_{h^\pm} dA_{h^\pm} = -4\pi(g^\pm - 1)\mu^\pm$ . Using this and Proposition 4.6,

$$(5.4) \quad \begin{aligned} -\frac{1}{12\pi} \int_{R_t^\pm} dA_{h^\pm} 2K_{h^\pm} \sigma_t^\pm &= \frac{2}{3}(g^\pm - 1) \int_{R_t^\pm} \mu^\pm \{ (g^\mp/g)^2 \log |t| - 2(g^\mp/g) \log G^\pm(z, p^\pm) \} + o(1) \\ &= \frac{2}{3}(g^\pm - 1)(g^\mp/g)^2 \log |t| + o(1) \end{aligned}$$

where we have also used the normalization (4.3). For the next term in (2.4), use the estimates in Proposition 4.6 and the fact, again from (4.5) and also (5.3), that

$$(5.5) \quad dA_{h^\pm} \Delta \sigma_t^\pm = 4\pi(g^\mp/g)\mu^\pm + o(1)$$

where  $o(1)$  converges pointwise to zero and is uniformly bounded on  $R_t^\pm$  with respect the euclidean measure in the plumbing coordinates. By (4.7), (4.8), and (5.5), we have

$$(5.6) \quad \begin{aligned} -\frac{1}{12\pi} \int_{R_t^\pm} dA_{h^\pm} |\nabla \sigma_t^\pm|^2 &= \frac{1}{12\pi} \int_{R_t^\pm} dA_{h^\pm} \sigma_t^\pm \Delta \sigma_t^\pm - \frac{1}{12\pi} \int_{\partial R_t^\pm} ds_{h^\pm} \sigma_t^\pm \partial_n \sigma_t^\pm \\ &= \frac{1}{12\pi} \int_{R_t^\pm} dA_{h^\pm} \sigma_t^\pm \Delta \sigma_t^\pm \\ &\quad + \frac{1}{12\pi} \int_{\partial R_t^\pm} ds_{h^\pm} \{ (g^+g^-/g^2) \log |t| + o(1) \} \partial_n \sigma_t^\pm \\ &= \frac{1}{12\pi} \int_{R_t^\pm} dA_{h^\pm} \Delta \sigma_t^\pm \{ (g^\mp/g)^2 \log |t| - 2(g^\mp/g) \log G^\pm(z, p^\pm) \\ &\quad + (g^+g^-/g^2) \log |t| \} + o(1) \\ &= \frac{1}{3} \left( \frac{g^\mp}{g} \right)^2 \log |t| + o(1) \end{aligned}$$

Since the metrics  $h^\pm$  are normalized at  $p^\pm$ ,

$$(5.7) \quad -\frac{1}{6\pi} \int_{\partial R_t^\pm} ds_{h^\pm} k_{h^\pm} \sigma_t^\pm = -\frac{1}{3}(g^+g^-/g^2) \log |t| + o(1) \quad (\text{by (4.7)})$$

Finally,

$$(5.8) \quad -\frac{1}{4\pi} \int_{\partial R_t^\pm} ds_{h^\pm} \partial_n \sigma_t^\pm = -\frac{1}{4\pi} \int_{R_t^\pm} dA_{h^\pm} \Delta \sigma_t^\pm = -\frac{g^\mp}{g} + o(1) \quad (\text{by (5.5)})$$

Combining the results (5.4) and (5.6)-(5.8) for  $R_t^+$  and  $R_t^-$ , we obtain the statement of the lemma.  $\square$

*Proof of Proposition 5.1.* It suffices to consider the factors in Proposition 2.5. By Corollary 3.6,

$$\frac{\ell_{h_t}(\gamma_t) \det \mathcal{N}_t^+ \det \mathcal{N}_t^-}{\ell_{h^+}(\gamma_t) \ell_{h^-}(\gamma_t) \det \mathcal{N}_t} \longrightarrow \frac{1}{2}$$

By [Wb, eq. (28)],

$$(\mathcal{Q}_{B_t})^2 = 2^{2/3} (2\pi)^{-1} |t|^{-1/3} \exp(-4\zeta'(-1) - 5/6)$$

The result now follows from Lemma 5.2.  $\square$

*Proof of Theorem 1.3.* Comparing Proposition 5.1 with Theorem 1.7 and the definition (1.1) of  $c_g$ , we find:  $c_{g+1} - c_g = c_1 - c_0$ , where  $c_1$  is the constant for  $g = 1$  surfaces and  $c_0$  is as in Proposition 5.1. The  $g = 1$  constant  $c_1$  has been evaluated (cf. [Fl, P]):  $c_1 = -8 \log(2\pi)$ . The expression for  $c_g$  now follows.  $\square$

## APPENDIX

At the suggestion of the referee, in this appendix we clarify the relationship between the expansions for holomorphic abelian differentials on degenerating surfaces found in [F1] and [Y]. Let  $\omega_t(z) \rightarrow \omega_0(z)$  be as in the statement of Lemma 4.9. We assume moreover that the periods  $\int_{A_j} \omega_t$  are fixed independent of  $t$ .

From (4.10) and (4.11) we have

$$(5.9) \quad W_0^+(z^+) = f(0, z^+)$$

$$(5.10) \quad \dot{W}_0^+(z^+) = \frac{h(0)}{z^+} + \partial_1 f(0, z^+) + \frac{\partial_2 f(0, z^+)}{z^+}$$

$$(5.11) \quad \ddot{W}_0^+(z^+) = 2 \frac{h'(0)}{z^+} + \partial_1^2 f(0, z^+) + 2 \frac{\partial_1 \partial_2 f(0, z^+)}{z^+} + \frac{\partial_2^2 f(0, z^+)}{(z^+)^2} + \frac{2}{(z^+)^2} g(0, z^+)$$

$$(5.12) \quad \dot{W}_0^-(z^-) = -\frac{h(0)}{z^-} - \frac{f(0, z^-)}{(z^-)^2} - g(0, z^-)$$

where the dots indicate derivatives with respect to  $t$ . From (5.10), the restriction of

$$\dot{\omega}_0(z) = \lim_{t \rightarrow 0} \frac{\omega_t(z) - \omega_0(t)}{t}$$

to  $M^+$  is an abelian differential with a pole of order at most one at  $p^+$ . Hence, it is holomorphic. On the other hand, by the normalization it must have zero  $A_j$  periods. It therefore vanishes identically. From (5.12), the restriction of  $\dot{\omega}_0$  to  $M^-$  is an abelian differential with a pole of order two at  $p^-$ , and the coefficient of the  $1/(z^-)^2$  term is  $-f(0, 0) = -W_0^+(0)$ , from (5.9). Hence, for  $z \in M^-$  we have the  $t$ -expansion

$$(5.13) \quad \omega_t(z) = -t W_0^+(0) \omega^-(z, p^-) + O(t^2)$$

where  $\omega^-(z, w)$  is the abelian differential of the second kind on  $M^-$ , normalized to have zero  $A_j$ -periods and an expansion

$$\omega^-(z, w) = \left( \frac{1}{(z-w)^2} + \frac{1}{6}S^-(p) + \dots \right) dzdw$$

in local coordinates about a point  $p$ , where  $S^-$  is a holomorphic projective connection on  $M^-$  (see [F1, Cor. 2.6]). Examining the constant term in the  $t$ -expansion (5.12) we find

$$\frac{1}{2}\partial_2^2 f(0, 0) + g(0, 0) = \frac{1}{6}W_0^+(0)S^-(p^-)$$

From (5.11), the restriction of  $\check{\omega}_0$  to  $M^+$  is an abelian differential with a pole of order two at  $p^+$ , and the coefficient of the  $1/(z^+)^2$  term is

$$\partial_2^2 f(0, 0) + 2g(0, 0) = \frac{1}{3}W_0^+(0)S^-(p^-)$$

Hence, for  $z \in M^+$  we have the  $t$ -expansion

$$(5.14) \quad \omega_t(z) = \omega_0(z) + \frac{t^2}{6}W_0^+(0)S^-(p^-)\omega^-(z, p^-) + O(t^3)$$

The expressions (5.13) and (5.14) agree with [Y, eq. (36)]. In [F1, eq. 47], the  $t$ -expansion was carried out with respect to a local expression of the differentials in the “pinching coordinate”  $\mathcal{X}$ . Since the restrictions of  $d\mathcal{X}$  to  $M^\pm - \{p^\pm\}$  are differentials that themselves depend on  $t$  (see (4.16)), the  $O(t)$  terms calculated there are incorrect.

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