

ON THE BLOW-UP SET OF THE YANG-MILLS FLOW ON KÄHLER SURFACES

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ABSTRACT. The Yang-Mills flow on a Kähler surface with holomorphic initial data converges smoothly away from a singular set determined by the Harder-Narasimhan-Seshadri filtration of the initial holomorphic bundle.

1. INTRODUCTION

Let (X, ω) be a compact Kähler surface. Given a holomorphic bundle (E, D_0'') on X , let $\{E_{ij}\}$ denote the Harder-Narasimhan-Seshadri (double) filtration of E , and let $\text{Gr}(E) = \bigoplus_{ij} Q_{ij}$, where $Q_{ij} = E_{ij}/E_{i,j-1}$ are the stable successive quotients of the filtration (see [DW1] for more details). Then $\text{Gr}(E)$ is a torsion-free sheaf on X whose double dual $\text{Gr}(E)^{**}$ is locally free. Denote by \mathcal{Z}^{alg} the singularity set of $\text{Gr}(E)$, i.e. the support of the torsion sheaf $\text{Gr}(E)^{**}/\text{Gr}(E)$. Associated to each point $p \in \mathcal{Z}^{alg}$ is an integral multiplicity μ_p^{alg} (see (2) below).

Given a hermitian metric on E , let

$$(1) \quad \frac{\partial D}{\partial t} = -D^* F_D, \quad D(0) = D_0,$$

denote the Yang-Mills flow with initial condition given by the hermitian connection associated to D_0'' and H . Let D_∞ denote the Yang-Mills connection associated to the holomorphic vector bundle $\text{Gr}(E)^{**}$, i.e. the hermitian connection for the direct sum of Hermitian-Einstein metrics on the stable factors of $\text{Gr}(E)^{**}$.

For a sequence $t_j \rightarrow \infty$, it follows from the work of Uhlenbeck and Sedlacek (see [U1, Se]) that there exists a subsequence t_{j_k} , and a finite set of points \mathcal{Z}^{an} , depending a priori on the subsequence, such that $D_{t_{j_k}}$ converges (modulo real gauge transformations) weakly in $L_{1,loc}^p(X - \mathcal{Z}^{an})$ to a limiting Yang-Mills connection. Furthermore, by [U2] this connection extends to a Yang-Mills connection on a bundle E_∞ on X . Call this connection an *Uhlenbeck limit*. The weak $L_{1,loc}^p$ convergence can be improved to strong C_{loc}^∞ convergence using parabolic estimates (cf. [St, CS, HT]), and by the main theorem of [DW1] any Uhlenbeck limit can be identified with the Yang-Mills connection D_∞ on $\text{Gr}(E)^{**}$.

Associated to each point p in the blow-up set \mathcal{Z}^{an} is an integral multiplicity μ_p^{an} measuring the concentration of the curvature of the sequence of connections $D_{t_{j_k}}$ (see Lemma 5 below). The first

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goal of this note is to show that the analytic and algebraic singular sets coincide, along with their multiplicities.

Theorem 1. *Let $(E, D_0'') \rightarrow X$ be a holomorphic hermitian vector bundle over a compact Kähler surface, and let Z^{alg} denote the algebraic singular set as above. Modulo real gauge transformations, the Yang-Mills flow D_t with initial condition D_0 converges smoothly as $t \rightarrow \infty$ to the Yang-Mills connection D_∞ on the vector bundle $\text{Gr}(E, D_0'')^{**}$ away from Z^{alg} . Moreover, for any sequence $t_j \rightarrow \infty$ defining an Uhlenbeck limit with blow-up set Z^{an} , $Z^{an} = Z^{alg}$ as sets with $\mu_p^{an} = \mu_p^{alg}$ for all p .*

To state the second result of this paper, consider $E \rightarrow X$ as a fixed smooth hermitian complex vector bundle, and let \mathcal{A}^s (resp. \mathcal{A}^{ss}) denote the space of stable (resp. semistable) holomorphic structures on E with the smooth topology. Here, stability is in the sense of Mumford-Takemoto with respect to the Kähler class ω . Let $\overline{\mathcal{M}}$ denote the Uhlenbeck compactification of the space of equivalence classes of ω anti-self-dual connections on E . We have

Theorem 2. *The Yang-Mills flow defines a continuous map from the closure of \mathcal{A}^s in \mathcal{A}^{ss} to $\overline{\mathcal{M}}$.*

In the case of Riemann surfaces it was conjectured in [AB], and proved in [D, R], that the map from \mathcal{A}^{ss} to the minimum of the Yang-Mills functional is a deformation retract. It follows that the equivariant cohomology of \mathcal{A}^{ss} (by the action of the gauge group) is equal to the equivariant cohomology of the moduli space of projectively flat connections. In the case of a Kähler surface, the corresponding map fails to be a retraction due to the bubbling described above. Theorem 2 serves as a substitute. It would be interesting to use this result to obtain more information about the topology of $\overline{\mathcal{M}}$. This space, for arbitrary rank, has recently been extensively studied in [Kr].

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2. PRELIMINARIES

Let (X, ω) be a Kähler surface, normalized so that $\text{vol}(X) = 2\pi$. The ω -slope $\mu(E)$ of a torsion-free sheaf $E \rightarrow X$ is defined by

$$\mu(E) = \frac{\text{deg}(E)}{\text{rk}(E)} = \frac{1}{\text{rk}(E)} \int_X c_1(E) \wedge \omega .$$

The sheaf is called *stable* (resp. *semistable*) if $\mu(F) < \mu(E)$ (resp. \leq) for all subsheaves $F \subset E$ with $0 < \text{rk}(F) < \text{rk}(E)$. Following [DW1], if $E \rightarrow X$ is a holomorphic vector bundle there is an associated double filtration $\{E_{ij}\}$ of holomorphic vector bundles E_{ij} , called the Harder-Narasimhan-Seshadri filtration, which is defined as follows. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

be the Harder-Narasimhan filtration of E , where $Q_i = E_i/E_{i-1}$ is torsion-free, semistable, and $\mu(Q_i) > \mu(Q_{i+1})$ for $i = 1, \dots, \ell - 1$. Then for each i , let

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset \dots \subset E_{i,\ell_i} = E_i$$

be the Jordan-Hölder filtration, where the successive quotients $Q_{ij} = E_{ij}/E_{i,j-1}$ are torsion-free, stable, and $\mu(Q_{ij}) = \mu(Q_i)$, $j = 1, \dots, \ell_i$. Furthermore, the associated graded object

$$\mathrm{Gr}(E) = \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{\ell_i} Q_{ij} ,$$

is uniquely determined by the isomorphism class of E (see [DW1, Sec. 2.1] for details).

Let $\xi \rightarrow X$ be a torsion-free sheaf. Since X is a surface ξ^{**} is locally free. The *singular set* $\mathcal{Z}^{alg.}(\xi)$, defined to be the support of the torsion sheaf ξ^{**}/ξ , is a finite collection of points (cf. [Ko, Cor. V.5.15 and V.5.20]). For $p \in X$, define the *algebraic multiplicity* $\mu_p^{alg.}(\xi)$ of ξ at p to be the dimension of $(\xi^{**}/\xi)_p$ as a \mathbb{C} -vector space. Since the support is at isolated points, this may be reformulated as

$$(2) \quad \mu_p^{alg.}(\xi) = h^0(U, \xi^{**}/\xi) ,$$

where U is an open set chosen so that $U \cap \mathcal{Z}^{alg.} = \{p\}$ if $p \in \mathcal{Z}^{alg.}$ and empty otherwise. It follows that $\mu_p^{alg.}(\xi)$ is a non-negative integer that is strictly positive if and only if $p \in \mathcal{Z}^{alg.}$.

Definition 3. Let (E, D'') be a holomorphic vector bundle. Define the *algebraic singular set* $\mathcal{Z}^{alg.}$, to be the singular set of $\mathrm{Gr}(E, D'')$ with multiplicities as in (2).

Next, we briefly review the Yang-Mills flow and the main result of [DW1]. Let $E \rightarrow X$ be a smooth complex vector bundle of rank r with Chern classes $c_1(E)$ and $c_2(E)$. For later reference, recall that:

$$(3) \quad -\mathrm{ch}_2(E)[X] = \left\{ c_2(E) - \frac{1}{2}c_1^2(E) \right\} [X] = \frac{1}{8\pi^2} \int_X \mathrm{Tr} F_D \wedge F_D ,$$

where D is any connection on E with curvature F_D .

Fix a hermitian metric H on E and denote by $\mathcal{A}^{1,1}$ the space of integrable hermitian connections on E , i.e. those for which the curvature is of type $(1,1)$. Let \mathfrak{G} (resp. $\mathfrak{G}^{\mathbb{C}}$) denote the space of real (resp. complex) gauge transformations. The Yang-Mills functional is given by

$$\mathrm{YM}(D) = \int_X |F_D|^2 d\mathrm{vol}_\omega ,$$

and is invariant under the action of \mathfrak{G} . Its L^2 -gradient flow on $\mathcal{A}^{1,1}/\mathfrak{G}$ with initial condition D_0 is given by (1). By the results of [Do], solutions of (1) for $D_0 \in \mathcal{A}^{1,1}$ exist for all $t > 0$. Furthermore, using the main theorem of [DW1] we have

Theorem 4. *Given $D_0 \in \mathcal{A}^{1,1}$ and any sequence $t_j \rightarrow \infty$, let $D_j = D_{t_j}$. Then there is*

- (1) *a subsequence (also denoted j);*
- (2) *a finite subset $\mathcal{Z}^{an.} \subset X$ and a nonnegative integer valued multiplicity $\mu_p^{an.}$ which is positive if and only if $p \in \mathcal{Z}^{an.}$;*

- (3) a smooth hermitian vector bundle (E_∞, H_∞) on X , smoothly isomorphic to $\text{Gr}(E, D_0'')^{**}$;
- (4) a smooth Yang-Mills connection D_∞ on (E_∞, H_∞) such that D_∞'' induces a holomorphic structure isomorphic to $\text{Gr}(E, D_0'')^{**}$;
- (5) a smooth isometry

$$\tau : (E_\infty, H_\infty)|_{X - \mathcal{Z}^{an.}} \longrightarrow (E, H)|_{X - \mathcal{Z}^{an.}} ,$$

such that $\tau(D_j) \rightarrow D_\infty$ smoothly on compact subsets of $X - \mathcal{Z}^{an.}$;

- (6) the Yang-Mills energy densities (with respect to H_∞):

$$|F_{\tau(D_j)}|^2 dvol_\omega \longrightarrow |F_{D_\infty}|^2 dvol_\omega + \sum_{p \in \mathcal{Z}^{an.}} \mu_p^{an.} \delta_p ,$$

in the weak-* topology, where δ_p is the Dirac measure at p .

Note that by [DoKr, Proposition 6.2.14], $\|D_t^* F_{D_t}\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$; hence, any Uhlenbeck limit is Yang-Mills. The contribution of [DW1] was to identify the limiting connections with the Yang-Mills connection on $\text{Gr}(E, D_0'')^{**}$. The convergence stated in [DW1] was weakly in $L_{1,loc}^p(X - \mathcal{Z}^{an.})$, however this can be improved to local C^∞ convergence using parabolic estimates analogous to the ones developed by Struwe [St] for the harmonic map flow (cf. [CS, Sections 5 and 6] and also [HT]).

Lemma 5. *The analytic multiplicity in Theorem 4 is given by*

$$(4) \quad \mu_p^{an.} = \lim_{j \rightarrow \infty} \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \{ \text{Tr } F_{D_j} \wedge F_{D_j} - \text{Tr } F_{D_\infty} \wedge F_{D_\infty} \} .$$

where $B_\varepsilon(p)$ is a ball about p chosen so that $B_\varepsilon(p) \cap \mathcal{Z}^{an.} = \{p\}$ if $p \in \mathcal{Z}^{an.}$ and empty otherwise.

Proof. By [Ko, IV.4.5],

$$|F_{D_j}|^2 dvol_\omega = |\Lambda F_{D_j}|^2 dvol_\omega + \text{Tr } F_{D_j} \wedge F_{D_j} .$$

By [DW1, Lemma 2.17], ΛF_{D_j} converges in any L^p . Hence, the result follows. As in [DoKr, Sec. 4.4], $\mu_p^{an.}$ is a non-negative integer that is strictly positive if and only if $p \in \mathcal{Z}^{an.}$ (see also the discussion in the next section). \square

3. PROOF OF THEOREM 1

In order to prove Theorem 1 we have to show that for any Uhlenbeck limit $\mu_p^{an.} = \mu_p^{alg.}$ for all $p \in X$. The key result is the following

Lemma 6. *Let $E \rightarrow X$ be a complex vector bundle with connection D , and let $\xi \rightarrow X$ be a torsion-free sheaf. Fix $p \in X$. We assume there is a modification $\pi : \widehat{X} \rightarrow X$, supported away from p , and a sheaf $\widehat{\xi} \rightarrow \widehat{X}$ such that*

- (1) $\widehat{\xi} \simeq \xi$ in a neighborhood of p ;
- (2) $\widehat{\xi}$ restricted to $\widehat{X} - \{p\}$ is the sheaf of holomorphic sections of a vector bundle that is smoothly isomorphic to $\pi^* E$;
- (3) $\text{ch}(\pi^* E) = \text{ch}(\widehat{\xi})$.

For any connection D_∞ on ξ^{**} , and for $B_{2\varepsilon}(p)$ disjoint from the center of π , it follows from (1) and (2) that we may write $D = D_\infty + a$ on $B_\varepsilon(p) - \{p\}$. Then we have

$$\begin{aligned} \mu_p^{alg.}(\xi) &= \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \{ \text{Tr } F_D \wedge F_D - \text{Tr } F_{D_\infty} \wedge F_{D_\infty} \} \\ &\quad + \frac{1}{8\pi^2} \int_{\partial B_\varepsilon(p)} \text{Tr} \left\{ a \wedge D_\infty(a) + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_{D_\infty} \right\}. \end{aligned}$$

Proof. By (1) and (2), ξ is locally free on $B_\varepsilon(p) - \{p\}$, and $\hat{\xi}$ is locally free on $\hat{X} - \{p\}$. Hence, if χ denotes the holomorphic Euler characteristic, it follows from the Hirzebruch-Riemann-Roch Theorem [OTT] that

$$\begin{aligned} \mu_p^{alg.}(\xi) &= h^0(B_\varepsilon(p), \xi^{**}/\xi) = h^0(B_\varepsilon(p), \hat{\xi}^{**}/\hat{\xi}) = h^0(\hat{X}, \hat{\xi}^{**}/\hat{\xi}) = \chi(\hat{\xi}^{**}/\hat{\xi}) = \chi(\hat{\xi}^{**}) - \chi(\hat{\xi}) \\ (5) \quad &= \int_{\hat{X}} \left\{ \text{ch}_2(\hat{\xi}^{**}) - \text{ch}_2(\hat{\xi}) \right\} = \int_{\hat{X}} \left\{ \text{ch}_2(\hat{\xi}^{**}) - \text{ch}_2(\pi^*E) \right\} \quad \text{by (3)}. \end{aligned}$$

Now D induces a connection \hat{D} on π^*E . Extend the connection D_∞ over $B_\varepsilon(p)$ smoothly to a compatible connection \hat{D}_0 on $\hat{\xi}^{**}$. Then from (3) and (5),

$$\begin{aligned} \mu_p^{alg.}(\xi) &= \frac{1}{8\pi^2} \int_{\hat{X}} \left\{ \text{Tr } F_{\hat{D}} \wedge F_{\hat{D}} - \text{Tr } F_{\hat{D}_0} \wedge F_{\hat{D}_0} \right\} \\ &= \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \text{Tr } F_D \wedge F_D - \text{Tr } F_{D_\infty} \wedge F_{D_\infty} \right\} + \frac{1}{8\pi^2} \int_{\hat{X} - B_\varepsilon(p)} \left\{ \text{Tr } F_{\hat{D}} \wedge F_{\hat{D}} - \text{Tr } F_{\hat{D}_0} \wedge F_{\hat{D}_0} \right\}. \end{aligned}$$

Now by the assumption that $\hat{\xi}$ and π^*E are smoothly isomorphic on $\hat{X} - B_\varepsilon(p)$, the second term on the right hand side can be transferred to the boundary via the secondary characteristic class. \square

Proposition 7. *Let D_j be a sequence along the Yang-Mills flow for an initial integrable connection D_0 on E , and let $\mathcal{Z}^{alg.}$ denote the algebraic singular set of the associated Harder-Narasimhan-Seshadri filtration of (E, D_0'') . Suppose there is an Uhlenbeck limit $D_j \rightarrow D_\infty$ with bubbling set $\mathcal{Z}^{an.}$. Then $\mathcal{Z}^{an.} = \mathcal{Z}^{alg.}$, with multiplicities.*

Proof. Write $a_j = D_j - D_\infty$ on $X - \mathcal{Z}^{an.}$. For any $p \in X$, choose ε sufficiently small such that $B_\varepsilon(p)$ intersects $\mathcal{Z}^{an.}$ and $\mathcal{Z}^{alg.}$ in at most $\{p\}$. Given (E, D_0'') with Harder-Narasimhan-Seshadri filtration $\{E_{ij}\}$ we can perform a sequence of monoidal transformations $\pi : \hat{X} \rightarrow X$, supported away from p and with exceptional divisor \mathbf{e} , such that π^*E has a double filtration $\{\hat{E}_{ij}\}$, where \hat{E}_{ij} are *subbundles* except possibly at p . We can also arrange that $\pi(\mathbf{e}) = \mathcal{Z}^{alg.} - \{p\}$. The proof of this statement follows from a slight modification of [DW1, Prop. 3.7] (see also [Bu1]). Note that $\{\hat{E}_{ij}\}$ is not necessarily the Harder-Narasimhan-Seshadri filtration of π^*E for any Kähler metric on \hat{X} . With this understood, set $\xi = \text{Gr}(E, D_0'')$, and let $\hat{\xi}$ be the associated graded object for the filtration $\{\hat{E}_{ij}\}$ of $\pi^*(E)$. Then $\hat{\xi}$ is smoothly isomorphic to $\pi^*(E)$ away from p . Applying Lemma

6 to E with connections D_j and ξ with connection D_∞ on ξ^{**} , we have

$$\begin{aligned} \mu_p^{alg.} &= \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \{ \text{Tr } F_{D_j} \wedge F_{D_j} - \text{Tr } F_{D_\infty} \wedge F_{D_\infty} \} \\ &\quad + \frac{1}{8\pi^2} \int_{\partial B_\varepsilon(p)} \text{Tr} \{ a_j \wedge D_\infty(a_j) + \frac{2}{3} a_j \wedge a_j \wedge a_j + 2a_j \wedge F_{D_\infty} \} . \end{aligned}$$

By (4), the limit of the first term on the right hand side as $j \rightarrow \infty$ is $\mu_p^{an.}$, and by Theorem 4 (5) the limit of the second term vanishes. Hence, $\mu_p^{alg.} = \mu_p^{an.}$ for all $p \in X$. This completes the proof. \square

Proof of Theorem 1. By Proposition 7, the bubbling set for any Uhlenbeck limit $D_{t_j} \rightarrow D_\infty$ is fixed. In particular, $\|F_{D_t}\|_{L^2(B_r(x))}$ is uniformly small on small balls $B_r(x)$ in the complement of $Z^{alg.}$. Indeed, if this were not the case then for a given $\varepsilon > 0$ we could find a sequence of radii $r_j \rightarrow 0$ and times $t_j \rightarrow \infty$ such that

$$\int_{B_{r_j}(x)} |F_{D_{t_j}}|^2 dvol \geq \varepsilon .$$

According to Theorem 4 (6), this would force x to be in the singular set of any subsequential Uhlenbeck limit of D_{t_j} , contradicting the assumption that $x \notin Z^{alg.}$. By [Do, Prop. 16] the Hermitian-Einstein tensors of D_t are uniformly bounded. On the other hand, the ε -regularity result in [UY, Theorem 5.1] and [U3, Theorems 2.2 and 3.5], which is a generalization of the result in [U2], states that there are positive constants c_1, c_2 and ε_0 such for any connection D satisfying $\mathcal{E}_D(x, r) < \varepsilon_0$ for r sufficiently small, where

$$\mathcal{E}_D(x, r) = c_1 r^4 \|\Lambda F_D\|_{L^\infty}^2 + \int_{B_r(x)} |F_D|^2 dvol ,$$

then we have

$$(6) \quad \sup_{B_{r/4}(x)} |F_D|^2 \leq \frac{c_2}{r^4} \mathcal{E}_D(x, r) .$$

Applying this to the flow, it follows that F_{D_t} is locally bounded away from $Z^{alg.}$ uniformly as $t \rightarrow \infty$. Then the local smooth convergence follows as discussed above (cf. [CS]). \square

4. PROOF OF THEOREM 2

Let D_i be a sequence of integrable unitary connections on $E \rightarrow X$ inducing stable holomorphic structures, and assume that $D_i \rightarrow D$ smoothly, where (E, D'') is semistable. Associated to (E, D'') is a Seshadri filtration by subsheaves and an algebraic singular set $Z^{alg.}$. Since D_i are stable, by the Donaldson's theorem [Do] the Yang-Mills flow with initial conditions D_i converges to an irreducible Hermitian-Yang-Mills connection \tilde{D}_i . By Uhlenbeck compactness, after passing to a subsequence, there is an Uhlenbeck limit $(E, \tilde{D}_i) \rightarrow (\tilde{E}_\infty, \tilde{D}_\infty)$ with a bubbling set $\tilde{Z}^{an.}$. On the other hand, by Theorem 1, the Yang-Mills flow with initial condition D converges smoothly away from $Z^{alg.}$ to the Hermitian-Yang-Mills connection D_∞ on $E_\infty = \text{Gr}(E, D'')^{**}$. The proof of Theorem 2 is complete

if we show that $\tilde{D}_\infty = D_\infty$ and $\tilde{Z}^{an.} = Z^{alg.}$ (with multiplicity). The first statement follows from a modification of the argument in [DW1] and the second by an argument similar to the one used above.

Let $D_{i,t}$ and D_t denote the time t solutions to the YM flow with initial conditions D_i and D , respectively. Fix $t_j \rightarrow \infty$ with $\|D_{t_j}^* F_{D_{t_j}}\|_{L^2} \rightarrow 0$. By Theorem 1 it follows that any subsequential Uhlenbeck limit of D_{t_j} converges to D_∞ . For each t_j we may choose i_j sufficiently large so that $\|D_{i_j, t_j} - D_{t_j}\| \rightarrow 0$ in any norm. In particular, any subsequential Uhlenbeck limit of D_{i_j, t_j} also converges to D_∞ .

With this understood, relabel the sequences $D_j = D_{i_j, t_j}$, $\tilde{D}_j = \tilde{D}_{i_j}$. Then the first claim is a consequence of the following general uniqueness result.

Lemma 8. *Let $D_j \rightarrow D_\infty$, $\tilde{D}_j = g_j(D_j) \rightarrow \tilde{D}_\infty$ converge weakly in $L^p_{1,loc}$ away from a finite set of points $Z \subset X$. Assume a uniform bound on $\|\Lambda F_{\tilde{D}_j}\|_{L^\infty}$, and suppose that (E_∞, D_∞) and $(\tilde{E}_\infty, \tilde{D}_\infty)$ are Hermitian-Einstein connections. Then $(E_\infty, D_\infty) \simeq (\tilde{E}_\infty, \tilde{D}_\infty)$.*

Proof. Since Hermitian-Einstein metrics are unique, it suffices to prove that (E_∞, D''_∞) and $(\tilde{E}_\infty, \tilde{D}''_\infty)$ are holomorphically isomorphic. Let S be an irreducible factor of (E_∞, D_∞) and π_S the projection to S . Let U be a finite union of disjoint balls containing Z , and set $X^* = X - \bar{U}$. Let $\tau_j = \text{Tr}(g_j^* g_j \pi_S)$, and suppose g_j has been rescaled so that

$$(7) \quad \int_{X^*} \tau_j^2 d\text{vol}_X = 1 .$$

Fix a ball $B_R \subset X^*$. By the main result of [DW2], we may find local holomorphic frames for $(E, D''_j)|_{B_R}$ converging in C^1 to a holomorphic frame for $(E, D''_\infty)|_{B_R}$. In particular, there exist complex gauge transformations φ_j on B_R such that

- (i) $\varphi_j \rightarrow \mathbf{I}$ uniformly;
- (ii) $g_j \varphi_j \pi_S : S|_{B_R} \rightarrow (E, \tilde{D}''_j)|_{B_R}$ is holomorphic.

Setting $\hat{\tau}_j = \text{Tr}((g_j \varphi_j)^* g_j \varphi_j \pi_S)$, it follows that (cf. [Do, p. 23])

$$\left(\sup_{B_r} \hat{\tau}_j \right)^2 \leq C_r \int_{B_R} \hat{\tau}_j^2 d\text{vol}_X ,$$

for any $r < R$. By (7) and (i), the right hand side above is uniformly bounded. Hence, again referring to the argument in [Do], we can extract a subsequence of $g_j \varphi_j \pi_S$ converging to a holomorphic map on B_R . By (i), it follows that $g_j \pi_S$ itself converges to this limit. By a diagonalization argument we conclude there is a subsequence, also denoted by $g_j \pi_S$, converging uniformly on compact subsets of X^* to a holomorphic map $g_\infty : S|_{X^*} \rightarrow (\tilde{E}_\infty, \tilde{D}''_\infty)|_{X^*}$, which then extends to X by Hartogs' Theorem. By the argument in [Bu2, Lemma 2.2], g_∞ is nonzero. Now since S is stable and $(\tilde{E}_\infty, \tilde{D}''_\infty)$ is polystable with the same slope as S , it follows that g_∞ is an isomorphism onto one of the irreducible factors of $(\tilde{E}_\infty, \tilde{D}''_\infty)$.

Assume that we have found in this way an isomorphism of $S_{(k)} = S_1 \oplus \cdots \oplus S_k \subset (E_\infty, D''_\infty)$ with $\tilde{S}_{(k)} = \tilde{S}_1 \oplus \cdots \oplus \tilde{S}_k \subset (\tilde{E}_\infty, \tilde{D}''_\infty)$. Let \tilde{Q} denote the orthogonal complement of $\tilde{S}_{(k)}$, and suppose

S_{k+1} is an irreducible factor of (E_∞, D''_∞) in the complement of $S_{(k)}$. By the previous construction, $(\pi_{\tilde{Q}})g_j\pi_{S_{k+1}}$ is injective for j sufficiently large (again identifying sequences and subsequences). We now proceed as above. Set $\tau_j = \text{Tr}(g_j^*(\pi_{\tilde{Q}})g_j\pi_{S_{k+1}})$ with the same normalization (7). On B_R we find complex gauge transformations φ_j, ψ_j such that

- (iii) $\varphi_j \rightarrow \mathbf{I}$ and $\psi_j \rightarrow \mathbf{I}$ uniformly;
- (iv) $(\pi_{\tilde{Q}})\psi_j g_j \varphi_j \pi_{S_{k+1}} : S_{k+1}|_{B_R} \rightarrow (\tilde{Q}, \tilde{D}''_\infty|_{\tilde{Q}})|_{B_R}$ is holomorphic.

As above we conclude the existence of a subsequence of $(\pi_{\tilde{Q}})g_j\pi_{S_{k+1}}$ converging uniformly on compact subsets of X^* to a nonzero holomorphic map $S_{k+1} \rightarrow (\tilde{Q}, \tilde{D}''_\infty|_{\tilde{Q}})$. Since S_{k+1} is stable and $(\tilde{Q}, \tilde{D}''_\infty|_{\tilde{Q}})$ is polystable with the same slope, S_{k+1} is isomorphic to an irreducible factor $\tilde{S}_{k+1} \subset \tilde{Q}$. Hence, we obtain an isomorphism $S_{(k+1)} \simeq \tilde{S}_{(k+1)}$, completing the inductive step and the proof of the lemma. \square

We will also require the following

Lemma 9. *Let $D_j \rightarrow D_\infty$ be an Uhlenbeck limit for a sequence of integrable holomorphic structures on E with bubbling set $\mathcal{Z}^{an.}$, and write $a_j = D_j - D_\infty$ on $X - \mathcal{Z}^{an.}$. Furthermore, we assume that the Hermitian-Einstein tensors ΛF_{D_j} are uniformly bounded in j . Then away from $\mathcal{Z}^{an.}$,*

- (1) $a_j \rightarrow 0$ in C_{loc}^α for any $0 < \alpha < 1$;
- (2) $D_\infty a_j$ is locally bounded, uniformly in j .

Proof. Since $a_j \rightarrow 0$ weakly in $L_{1,loc}^p$ for any p , the first statement follows from Sobolev embedding. For the second statement, note that

$$D_\infty a_j = F_{D_j} - F_{D_\infty} - a_j \wedge a_j .$$

By the assumption on Hermitian-Einstein tensors and the ε -regularity theorem [U3], F_{D_j} is locally bounded away from $\mathcal{Z}^{an.}$ (cf. (6)). The result follows. \square

Proof of Theorem 2. Choose $p \in X$ and ε so small that $B_{2\varepsilon}(p)$ intersects $\tilde{\mathcal{Z}}^{an.} \cup \mathcal{Z}^{alg.}$ in at most p . On $B_\varepsilon(p) - \{p\}$, we may define: $D_\infty = D_T + a_T$ and $D_j^\infty = D_T + a_{j,T}$. We assume T has been chosen sufficiently large so that the following two assumptions hold. First,

$$(8) \quad \left| \mu_p^{alg.} - \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \{ \text{Tr } F_{D_T}^2 - \text{Tr } F_{D_\infty}^2 \} \right| < \frac{1}{4} .$$

This is possible by Proposition 7. Second, by Theorem 1, we may assume for large T that

$$(9) \quad \left| \frac{1}{8\pi^2} \int_{\partial B_\varepsilon(p)} \text{Tr} \{ a_T \wedge D_T(a_T) + \frac{2}{3} a_T \wedge a_T \wedge a_T + 2a_T \wedge F_{D_T} \} \right| < \frac{1}{4}$$

With this understood, we now show $\tilde{\mu}_p^{an.} = \mu_p^{alg.}$. The connections D_j^∞ are Hermitian-Einstein; hence, it follows as in the proof of Lemma 5 that

$$\begin{aligned} \tilde{\mu}_p^{an.} &= \lim_{j \rightarrow \infty} \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \text{Tr } F_{D_j^\infty}^2 - \text{Tr } F_{\bar{D}_\infty}^2 \right\} \\ &= \lim_{j \rightarrow \infty} \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \text{Tr } F_{D_j^\infty}^2 - \text{Tr } F_{D_\infty}^2 \right\} \quad \text{by Lemma 8 ,} \\ &= \lim_{j \rightarrow \infty} \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \text{Tr } F_{D_j^\infty}^2 - \text{Tr } F_{D_T}^2 \right\} + \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \text{Tr } F_{D_T}^2 - \text{Tr } F_{D_\infty}^2 \right\} . \end{aligned}$$

Now D_j^∞ and D_T are connections on the same bundle. Hence,

$$\begin{aligned} \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \text{Tr } F_{D_j^\infty}^2 - \text{Tr } F_{D_T}^2 \right\} &= \\ &= \frac{1}{8\pi^2} \int_{\partial B_\varepsilon(p)} \text{Tr} \left\{ a_{j,T} \wedge D_T(a_{j,T}) + \frac{2}{3} a_{j,T} \wedge a_{j,T} \wedge a_{j,T} + 2a_{j,T} \wedge F_{D_T} \right\} . \end{aligned}$$

By Lemma 9, we may assume that $a_{j,T} \rightarrow a_T$ in C_{loc}^α and that $D_T(a_{j,T})$ is locally bounded, uniformly in j , in $B_{2\varepsilon}(p) - \{p\}$. Letting $j \rightarrow \infty$, we deduce that

$$\begin{aligned} \tilde{\mu}_p^{an.} &= \frac{1}{8\pi^2} \int_{B_\varepsilon(p)} \left\{ \text{Tr } F_{D_T}^2 - \text{Tr } F_{D_\infty}^2 \right\} \\ &\quad + \frac{1}{8\pi^2} \int_{\partial B_\varepsilon(p)} \text{Tr} \left\{ a_T \wedge D_T(a_T) + \frac{2}{3} a_T \wedge a_T \wedge a_T + 2a_T \wedge F_{D_T} \right\} . \end{aligned}$$

By (8) and (9), $|\tilde{\mu}_p^{an.} - \mu_p^{alg.}| < 1/2$. Since the multiplicities are integers, they must coincide. \square

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