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Chapter 1
Algebra in \( \mathbb{R}^n \)

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Generalization to \( \mathbb{R}^n \)

Chapter 3
Generalization to \( \mathbb{R}^n \)

Once we get past this, we can do more practical examples.

Chapter 7, 8, 9.
What did we do?

We wanted to generalize properties of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) to \( \mathbb{R}^n \).

We introduced the notion of vector space, which generalized this.

Then we noted that \( \mathbb{R}^n \) is a vector space.

What are other examples of vector spaces which are in \( \mathbb{R}^n \)?

We defined a subspace, which is nothing but a vector space itself or something which is inside \( \mathbb{R}^n \).

eg. a line through \((0,0)\) in \( \mathbb{R}^2 \)

a plane passing through \((0,0,0)\) in \( \mathbb{R}^3 \)

We discussed that if \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a linear transformation, then the kernel of \( T = \{ x \in \mathbb{R}^n / T(x) = 0 \} \) and
Image \( T = \{ \vec{y} \in \mathbb{R}^m / \vec{y} = Tx \} \) for some \( x \in \mathbb{R}^n \)

\( x \) and \( y \) are subspaces of \( \mathbb{R}^n \) and \( \mathbb{R}^m \) respectively.

To check if something is a subspace of a vector space, because it sits inside a bigger space we only need to check 3 properties.

Eq: \( W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 / x_1 + x_2 + x_3 = 0 \right\} \)

is subspace because:

\[ 0 + 0 + 0 = 0 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W \]

For any \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \), \( y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in W \) \( \Rightarrow x_1 + x_2 + x_3 = 0 \)

\( y_1 + y_2 + y_3 = 0 \)

\[ x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} \in W, \quad \Leftrightarrow \quad (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = 0 \]
Similarly, \( k \in \mathbb{R}, \ x \in \mathbb{W} \) then
\[
\begin{align*}
x_1 + x_2 + x_3 &= 0 \\
\Rightarrow& \ k(x_1 + x_2 + x_3) = 0 \\
\Rightarrow& \ k \in \mathbb{R}, \ k(x_1 + kx_2 + kx_3) = 0
\end{align*}
\]
\(\Rightarrow\) \( k \cdot x \in \mathbb{W} \).

Alternatively, if we note that
\[
\mathbb{W} = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \}
\]

then
\[
\mathbb{W}' = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \}
\]

\(=\) \( \ker [1 \ 1 \ 1] \)

or it is the kernel of the transformation \( T: \mathbb{R}^3 \to \mathbb{R}^3 \) defined as
\[
\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = [1 \ 1 \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

is \( \mathbb{W}' = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1 \} \)

a subspace of \( \mathbb{R}^3 \)?
Given any vectorspace, it can be described completely by its basis.  
For example.

If \[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} \in W
\]

Then \[x_1 + x_2 + x_3 = 0\]

\(\Rightarrow\) Let \(x_2 = s\), \(x_3 = t\) for \(s, t \in \mathbb{R}\)

Then \(x_1 = -s - t\)

\(\Rightarrow\) \[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  -s-t \\
  s \\
  t
\end{bmatrix} = s \begin{bmatrix}
  -1 \\
  1 \\
  0
\end{bmatrix} + t \begin{bmatrix}
  -1 \\
  0 \\
  1
\end{bmatrix}
\]

\(\Rightarrow\) \(W = \text{Span} \left\{ \begin{bmatrix}
  -1 \\
  0 \\
  1
\end{bmatrix}, \begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix} \right\} \)

Further look at, \(\left\{ \begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}, \begin{bmatrix}
  0 \\
  0 \\
  1
\end{bmatrix} \right\} \). It is

is a linearly independent.
One can note this by observing that they are not scalar multiples of each other on that
\[ a_1 \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
\[ \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 6 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]
\[ \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
\[ \Rightarrow a_1 = 0 = a_2 \]
\[ \Rightarrow \text{Ker} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ is a basis of } W, \]

What is so special about the basis?
THM: Let \( V \) be a vector space (or a subspace) and \( \{v_1, v_2, \ldots, v_k\} \) \( \subseteq V \). Then \( \{v_1, v_2, \ldots, v_k\} \)
is a basis of $V$ if and only if
for every element $v \in V$ can be written uniquely as a linear combination
of $v_1, \ldots, v_k$, that is,

$$v = a_1 v_1 + \cdots + a_k v_k$$

for some $v_1, \ldots, v_k \in \mathbb{R}$.

This means that if I fix a basis and $\{v_1, \ldots, v_k\} \subseteq V$
and write down a $k$-tuple of real numbers $[c_1 \ldots c_k]$, then there can
be only one element of $V$ described by these scalars.

$$v = a_1 v_1 + \cdots + a_k v_k$$

We will see how this can be useful in a bit. But before that,

note $V$ can have more than one basis.