This consistent with our earlier 80 statements for $\mathbb{R}^2$ and $\mathbb{R}^3$.

(i) Any set of two non-parallel vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$.

(ii) Any set of three non-planar vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$.

How is having these different bases help us?

For example, consider this situation:

We have a playing field (rectangular) so it is like $\mathbb{R}^2$.loid, Noreen and Rachel are practicing throwing javeline.
However, they just have one Javelin to play with.

They need to record the spots where they are throwing the Javelin.

Now there are two non parallel running tracks close to the boundary of the field which are marked with distances for the runners.

How can they use this data to record their throws without a measuring tape?

Think of these two tracks as a basis of $\mathbb{R}^2$.

Now every point on the field can be written in terms of these two.
Let these be in the direction called 81°,

\[ \{ \vec{v}_1, \vec{v}_2 \} = \text{81°} \]

Then, we can expand the point

\[ \vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \]

\[ \downarrow \]

this is just an alternate notation.

Now, if we had known what \( \vec{v}_1, \vec{v}_2 \) are in terms of our regular basis \( \{ \vec{e}_1, \vec{e}_2 \} \) then we may be able to compute the length of \( \vec{v} \).

Example: Say \[ \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

Then

\[ \vec{v} = 2a_1 \vec{e}_1 + a_1 \vec{e}_2 + 2a_2 \vec{e}_1 + a_2 \vec{e}_2 = (2a_1 + a_2) \vec{e}_1 + a_1 \vec{e}_2 \]
In $\hat{e}_1, \hat{e}_2$ coordinates:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_B \rightarrow \begin{bmatrix} 2a_1 + a_2 \\ a_4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Notice what this is saying if $S = \{ \hat{e}_1, \hat{e}_2 \}$, $B = \{ \hat{v}_1, \hat{v}_2 \}$ and then transferring from $B$ coordinates to $S$ coordinates is given by the matrix:

$$S_A = \begin{bmatrix} [\hat{v}_1]_S & [\hat{v}_2]_S \end{bmatrix}$$

Try writing $S(\hat{e}_1, \hat{e}_2)$ in terms of $\hat{v}_1, \hat{v}_2$ to check that $\begin{bmatrix} [\hat{e}_1]_B & [\hat{e}_2]_B \end{bmatrix} = S^{-1}$.
So far all the examples of vectorspace (subspaces) we have considered are subspaces of $\mathbb{R}^n$.

What are other possible examples?

1. Let $P_n = \{ f: \mathbb{R} \to \mathbb{R} \mid f(x) = a_0 + a_1 x + \ldots + a_n x^n \}$

   $\forall a_0, a_1, \ldots, a_n \in \mathbb{R}$

   Addition is termwise

   $(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)$

   $= (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n$

   Scalar multiplication is also termwise

   $k \cdot (a_0 + a_1 x + \ldots + a_n x^n) = k \cdot a_0 + k \cdot a_1 x + \ldots + k \cdot a_n x^n$

   One of course needs to check the eight properties.

   Let me skip to 3:

   0 polynomial is the polynomial with all coefficient being zero.
For any \((a_0 + a_1 x + \ldots + a_n x^n) + (0 + 0 x + \ldots + 0 x^n)\)

\[= a_0 + \ldots + a_n x^n \quad \text{for any} \quad a_0, a_1, \ldots, a_n \in \mathbb{R}\]

(4) For any \(a_0 + a_1 x + \ldots + a_n x^n\) and the polynomial

\[-a_0 + (-a_1)x + \ldots + (-a_n)x^n\]

is the inverse.

Check the other properties.

2) Consider the set of all \(m \times n\) real matrices, \(\mathbb{R}^{m \times n}\)

\[
\mathbb{R}^{m \times n} = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \quad \text{for all} \quad i = 1, \ldots, m \quad j = 1, \ldots, n \right\}
\]

We can define addition and scalar multiplication termwise.

1) \((A + B) + C = A + (B + C)\)

2) \(A + B = B + A\)

3) \(O\) matrix has all entries \(0\)

\(O + A = A = A + O\)

and so on.
3). Set of all real continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \).

\[ \mathcal{C}(\mathbb{R}/\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} / \text{f is continuous at all } a \in \mathbb{R} \} \]

Addition of functions:

\[ f, g : \mathbb{R} \to \mathbb{R}, \quad (f + g)(x) = f(x) + g(x) \]

Then, \( f + g \) is continuous.

\[ k \in \mathbb{R}, \quad (kf)(x) = kf(x) \]

and \( kf \) is continuous.

Zero function \( 0 : \mathbb{R} \to \mathbb{R} \)

\[ s + 0(x) = 0 \]

Constant function is continuous.

4). Set of all differentiable functions on \( \mathbb{R} \).

\[ \mathcal{D}(\mathbb{R}/\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} / \text{f is diff at } \} \]

is a subspace of \( \mathcal{C}(\mathbb{R}/\mathbb{R}) \)

5). In fact \( \mathcal{P}_n \subseteq \mathcal{D}(\mathbb{R}/\mathbb{R}) \subset \mathcal{C}(\mathbb{R}/\mathbb{R}) \)

is also a subspace.
6) The spectrum of infinite real sequences.

Let \( \{x_n \} / x_i \in \mathbb{R} \} = S \)

Then \( x + y = (x_1 + y_1, \ldots, x_n + y_n, \ldots) \)

Another sequence.

and \( kx = (kx_1, \ldots, kx_n, \ldots) \)

is a sequence.

One can think of this as \( \mathbb{R}^n \) as a subspace of this \( S \).

7) \( C^0(\mathbb{R}, \mathbb{R}) = \) infinitely many times differentiable real valued functions on \( \mathbb{R} \).

8) \( \mathbb{C} = \) complex numbers.

8) Examples:

Is \( \{ \begin{bmatrix} a+b & c \\ 0 & a+b \end{bmatrix} / a, b, c \in \mathbb{R} \} \subseteq M_{2 \times 2} \)

a subspace of \( M_{2 \times 2} \)?
8) Set of all invertible 2x2 matrices?

7) Polynomials of the form degree 2.
\[ f(x) = a_0 + a_2x^2 \] \( \mathbb{P}_2 \)?

6) \( \{ A \in \mathbb{M}_{2\times2} \mid \det A = 0 \} \)?

Allow definition of course still make sense.
In \( \mathbb{R}^2, \mathbb{R}^3 \), linear transformations corresponded to geometric operations which preserved linearity properties.
We can generalize this to compare subspaces of \( \mathbb{R}^n \) and these are completely described by matrices.
However, when we're thinking of more general vector spaces, \( L(T) \) is a way to compare these linearity properties.
Define \( W = \{ \begin{bmatrix} a & b \\ 0 & b \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \} \)

and \( P_2 \) be the vector space of all polynomials with degree \( \leq 2 \).

Then let \( L : W \rightarrow P_2 \)

\[
L \left( \begin{bmatrix} a & b \\ 0 & b \end{bmatrix} \right) = a + bx + bx^2.
\]

Is this a linear transformation?

\[
L \left( \begin{bmatrix} a & b \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & d \\ 0 & d \end{bmatrix} \right) = L \left( \begin{bmatrix} a+c \\ b+d \end{bmatrix} \right)
\]

\[
= (a+c) + (b+d)x + (b+d)x^2
\]

\[
L \left( \begin{bmatrix} a & b \\ 0 & b \end{bmatrix} \right) + L \left( \begin{bmatrix} c & d \\ 0 & d \end{bmatrix} \right) = a + bx + bx^2 + c + dx + dx^2
\]

L is a linear transformation.