For \( A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix} \) we cannot write it in the form \( \mathbf{P} \mathbf{D} \mathbf{P}^T \) because we do not have enough no. of \( \mathbf{1} \) in \( \mathbf{P} \) to have orthogonal eigenvectors.

Some simple facts about eigenvectors:

1) Let \( A \) be an \( n \times n \) matrix with characteristic polynomial

\[ f(\lambda) = (-\lambda)^n + b_{n-1} (-\lambda)^{n-1} + \ldots + b_0 \]

Then, \( b_{n-1} = \text{trace} \ A \).

\( b_0 = \det A \).

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det (A - \lambda I_2) = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \]

\[ = (a - \lambda)(d - \lambda) - bc \]

\[ = \lambda^2 - (a + d) \lambda + (ad - bc) \]

\[ \downarrow \quad \text{Trace of } A \quad \det A \]
2. If \( A = PBP^{-1} \) then for some \( n \times n \) invertible matrix \( P \) and \( B \) is a \( n \times n \) matrix.

Then \( A \) and \( B \) have the same eigenvalues but possibly different eigenvectors.

Definition: Let \( A \) and \( B \) are \( n \times n \) matrices such that \( A = PBP^{-1} \) for some invertible \( n \times n \) matrix. Then \( A \) and \( B \) are said to be \( \text{similar} \).

1) \( \det A = \det B \)
2) \( A \) is similar to \( B \) and then \( B \) is similar to \( A \).
3) \( A \) is similar to \( A \)
4) \( A \) is similar to \( B \) and \( B \) is similar to \( C \) then \( A \) is similar to \( C \).
5) \( \text{trace of } A = \text{trace of } B \).
Defn: A is said to be diagonalizable if A is similar to a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Thm: Let A be a $n \times n$ matrix such that geometric multiplicity of $\lambda_i$ equals algebraic multiplicity of $\lambda_i$.

Then A is diagonalizable.

Our first example had this property. For $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ we have two eigenvectors corresponding to eigenvalues 5 and -1; $\{ [1], [-1] \}$ which are linearly independent and therefore span $\mathbb{R}^2$.

Definition: If a matrix A has n linearly independent eigenvectors $\hat{v}_1, \ldots, \hat{v}_n$ corresponding to its eigenvalues.
Then \( \{v_1, \ldots, v_n\} \) is called an eigenbasis of \( \mathbb{R}^n \).

A is diagonalizable if and only if we can find an eigenbasis of \( \mathbb{R}^n \) corresponding to \( A \).

If \( \{\lambda_1, \ldots, \lambda_n\} \) are eigenvalues of \( A \) repeated up to multiplicity and \( \{v_1, \ldots, v_n\} \) is the corresponding eigenbasis,

\[
A = \begin{bmatrix}
\vdots & & \vdots \\
v_1 & \cdots & v_n \\
\vdots & & \vdots 
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & & 0 \\
0 & \cdots & 0 \\
0 & \cdots & \lambda_n
\end{bmatrix}
\begin{bmatrix}
\vdots \\
v_1 \\
\vdots
\end{bmatrix}
\]

Let us remind ourselves of our initial example.
We wanted to solve equations of the form
\[ \dot{x}(k) = A \dot{x}(k-1) \]
given some initial value \( \dot{x}(0) \).

Let \( A \) be diagonalizable. Then \( \{ \hat{v}_1, \ldots, \hat{v}_n \} \) is an eigenbasis of \( \mathbb{R}^n \). Therefore \( \dot{x}(0) = c_1 \hat{v}_1 + \cdots + c_n \hat{v}_n \).

Then, \( \dot{x}(1) = A \dot{x}(0) = c_1 A \hat{v}_1 + \cdots + c_n A \hat{v}_n \)
\[ = c_1 \lambda_1 \hat{v}_1 + \cdots + c_n \lambda_n \hat{v}_n \]

\[ \dot{x}(k) = c_1 \lambda_1^k \hat{v}_1 + \cdots + c_n \lambda_n^k \hat{v}_n \]

Then the behaviour of \( \lambda_1, \ldots, \lambda_n \) completely determines what happens to \( \dot{x}(k) \) as \( k \) becomes large.
For instance, if $\lambda_1, \ldots, \lambda_n < 1$

Then as $k \to \infty$, $\lambda_1^k, \ldots, \lambda_n^k \to 0$.

$\Rightarrow \quad \hat{x}(k) \to 0$

What is this $c_1, c_2, \ldots, c_n$?

$\hat{x}(0) = [v_1 \ldots v_n] \begin{bmatrix} c_1 \\ \\ \vdots \\ c_n \end{bmatrix}$

$P = [c_1 \ldots c_n]$

$\begin{bmatrix} c_1 \\ \\ \vdots \\ c_n \end{bmatrix} = P \hat{x}(0)$

(This is what you will get if instead you had replaced $A$ by $PDP^{-1}$.)
Let us now consider our owl and wood rat population.

After time \( t \) years

\[
\begin{bmatrix}
X(t) \\
W(t)
\end{bmatrix} = 
\begin{bmatrix} 0.5 & 0.4 \\ -p & 0.1 \end{bmatrix}
\begin{bmatrix}
X(t-1) \\
W(t-1)
\end{bmatrix}
\]

Let \( p = 0.02 \)

Then

\[
\det (A - \lambda I) = \det 
\begin{bmatrix} 0.5 - \lambda & 0.4 \\ -0.02 & 0.1 - \lambda \end{bmatrix}
\]

\[
= 0.55 + \lambda^2 - 1.6\lambda + 0.08
\]

\[
= \lambda^2 - 1.6\lambda + 0.63
\]

\[
= \lambda^2 - 0.9\lambda - 0.7\lambda + 0.63
\]

\[
= (\lambda - 0.9)(\lambda - 0.7)
\]

\[
\therefore \text{Eigenvalues of } A \text{ are } \lambda = 0.9, 0.7.
\]
Starting with some \( X(0) \), let \( V_1, V_2 \) be the eigenvectors corresponding to \( \lambda = 0.9 \) and \( \lambda = 0.7 \).

Then \( X(0) = 0.9V_1 c_1 V_1 + 0.7V_2 \) for some \( c_1, c_2 \in \mathbb{R} \).

And
\[
X(t) = c_1 0.9^t V_1 + c_2 0.7^t V_2
\]

As \( t \to \infty \) what happens?

Error: If we computed what \( V_1, V_2 \) were then this would give us a better idea of what how soon this ecosystem would disappear!