So far we have been only considering
discrete dynamical systems.
However, in several practical applications
the change happens instantaneously instead
of after discrete-time intervals!

For example, if we are considering
a prey-predator problem with animal
species which have relatively short-
life cycles, we care about instantaneous
changes!

A simple example. Suppose we
deposit \( P \) dollars in a bank which
gives 6% interest per year. Then
after a year we have
\[
T = P + \frac{6}{100} P = 1.06P.
\]
\[ P(T(2)) = P(1) \cdot (1.06)^2 \]
and so on.

Now, if we got the same interest compounded continuously, then if \( P(t) \) = amount in bank at time \( t \).

\[ \frac{dP(t)}{dt} = 0.06 \cdot P(t) \]

Now we want a solution this equation.

Do we know a function whose derivative is itself?

For example, \( P(t) = e^{0.06t} \)

Satisfies this equation.

But so does \( ce^{0.06t} \).
If we know \( P(0) \)
then \( P(0) = C \Rightarrow P(t) = P(0)e^{At} \)

Now if this was a linear system of first differential equations, i.e.,

\[
\frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t)
\]

\[
\frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t)
\]

Then we can write it as

\[
\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)
\]

Where \( \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \)

and \( A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \)

In the case that \( A \) is diagonal matrix,

\[
\frac{dx_1(t)}{dt} = \lambda_1 x_1(t)
\]

\[
\frac{dx_n(t)}{dt} = \lambda_n x_n(t)
\]
it is easy to solve for $x^j(t)$

$$x_1(t) = x_1(0) e^{kt}$$

$$x_n(t) = x_n(0) e^{nt}$$

$$\Rightarrow \quad \dot{x}(t) = x(0) e^{kt} \cdot e_1 + \cdots + x_n(0) e^{nt} e_n$$

When $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, \quad $e_n = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

are the standard vectors.

Example $\quad \frac{dx}{dt} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} x^j(t)$

$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ is not diagonal.

but let us see if diagonalizing helps
\[ f(x) = x^2 - 4x + 4 - 9 \]
\[ = x^2 - 5x + 1 = (x-1)(x-5) \]

When \( \lambda = 1 \)
\[
A + \lambda I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}
\]

\[ x_1 = -x_2 \]

3) Eigenspace of \( \lambda = 1 \)
\[ \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = 5 \]
\[
A - 5I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}
\]

\[ x_1 = x_2 \quad \text{Eigenspace of } \lambda = 5 \]

Then \( A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \]

\[
\begin{bmatrix} 1 \\ 1 \\ 5 \\ 5 \end{bmatrix} \quad \text{D} \quad \begin{bmatrix} 1 \\ 5 \end{bmatrix}^{-1}
\]
\[
\frac{dx}{dt} = A x(t) = SD S^d x(t)
\]

\[\frac{dS}{dt} = \sigma S h(t)\]

Define a new function
\[\hat{y}(t) = S^d x(t)\]

\[\frac{dy}{dt} = S^d \frac{dx}{dt}\]

\[\frac{dy}{dt} = D \hat{y}(t)\]

\[\hat{y}(t) = \begin{bmatrix}
y_1(0) e^{\lambda t} \\
y_2(0) e^{\mu t} \\
y_n(0) e^{\gamma t}
\end{bmatrix}\]

\[= \begin{bmatrix}
y_1(t) e^{-t} \\
y_2(t) e^{t}
\end{bmatrix}\]
Now \( y'(0) = S^{-1} x'(0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \)

and \( y(t) = S^{-1} x(t) \)

\( x(t) = S \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{st} \end{bmatrix} \)

\( = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{st} \end{bmatrix} \)

\( x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{st} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

\( = c_1 e^{-t} v_1 + c_2 e^{st} v_2 \)

Now what happens as \( t \to \infty \), as \( R^{-t} \to 0 \) but \( e^{st} \to \infty \).
Let \( c_1 = -1, c_2 \).

Then \( x(t) = \).

In general, let \( \frac{dx(t)}{dt} = A x(t) \)

for some matrix \( A \) which is diagonalizable.

Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( A \) with \( n \) linearly independent eigenvectors \( v_1, v_2, \ldots, v_n \).
Then, \( \dot{y}(t) = c_1 e^{\lambda_1 t} V_1 + \ldots + c_n e^{\lambda_n t} V_n \)

where \( \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix}^{-1} \begin{bmatrix} \ddot{y}(0) \\ \vdots \\ \ddot{y}(0) \end{bmatrix} \).

Corollary: When \( \lambda_i < 0 \) for all \( i = 1, 2, \ldots, n \)

Then \( \frac{d}{dt} \dot{y}(t) = A \dot{y}(t) \) has a

stable equilibrium at \( \ddot{y} = 0 \).

Now, what happens if \( A \) has complex eigenvalues?

Then we need to know how to solve

\[ \frac{dz}{dt} = \lambda z \quad \text{for} \ \lambda \in \mathbb{C}. \]
We are going to work with only a particular kind of complex valued functions

\[ z : \mathbb{R} \to \mathbb{C} \]

\[ z(t) = x(t) + iy(t) \]

Then note

\[ \frac{dz}{dt} = \lim_{h \to 0} \frac{z(t+h) - z(t)}{h} \]

\[ = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} + i \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \]

\[ = \frac{dx}{dt} + i \frac{dy}{dt} \]

And if we have \( f : \mathbb{R} \to \mathbb{R} \)

\[ f(z) \]

Then

\[ \frac{d}{dt} z(f(t)) = \frac{d}{ds} z(s) \cdot \frac{df}{dt} \]
Now we want to find $z$ such that \[ \frac{dz}{dt} = \lambda z. \]

Then: There exists a function $z: \mathbb{R} \rightarrow \mathbb{C}$ satisfying the equation \[ \frac{dz}{dt} = \lambda z(t). \]

Define this as $e^{\lambda t}$.

Now when $\lambda = i$

Let $z(t) = \cos t + i \sin t$.

Then \[ \frac{dz}{dt} = -\sin t + i \cos t = i (\cos t + i \sin t) = iz(t). \]

So \[ e^{it} = \cos t + i \sin t. \]
Further, one can check that

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Further, one can check that
Last class you learnt how to solve continuous dynamical systems.

\[
\frac{dx(t)}{dt} = \mathbf{A} \mathbf{x}(t)
\]

If \( \mathbf{A} \) is a \( n \times n \) matrix with \( n \) (complex) eigenvalues \( \lambda_1, \ldots, \lambda_n \) repeated up to multiplicity with \( n \) linearly independent eigenvectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) (as complex vectors in \( \mathbb{C}^n \)), then

\[
\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + \ldots + c_n e^{\lambda_n t} \mathbf{v}_n
\]

where

\[
\begin{bmatrix}
    c_1 \\
    \vdots \\
    c_n
\end{bmatrix} = \begin{bmatrix}
    \mathbf{v}_1 \\
    \vdots \\
    \mathbf{v}_n
\end{bmatrix}^{-1} \mathbf{x}(0)
\]

Clearly, if \( \text{Re} \lambda_i < 0 \) then \( \mathbf{x}(t) \to 0 \) as

\[
\text{for all } i = 1, 2, \ldots, n \quad t \to \infty
\]

P.T.O
When \( A \) is complex, \( x = p + i q \),
\[ e^{At} = e^{pt} (\cos qt + i \sin qt) \]

If \( p < 0 \) then \( e^{pt} \to 0 \) as \( t \to \infty \).

As before let us analyse the \( 2 \times 2 \) case for when we have real eigenvalues and when we have complex eigenvalues.

Let \( A \) be a \( 2 \times 2 \) matrix with eigenvalues \( \lambda_1, \lambda_2 \).

If \( \lambda_1, \lambda_2 \) are real,

Then 1) \( \lambda_1 \neq \lambda_2 \) \(
\implies \)
the system has a stable equilibrium.

\( \lambda < 0 < \lambda_1 \) then clearly the values will move toward \( \text{Span} \{v_1\} \) exponentially as \( t \to \infty \).
Case 3) \( 0 < \lambda_2 < \lambda_1 \)

as \( t \to \infty \), the values will lean towards

When \( \lambda_1, \lambda_2 \) are complex

\[ \lambda_1 = p + iq \quad \lambda_2 = p - iq \]

\[ \mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 \]

We know if \( p < 0 \) then \( \Theta \) is a stable equilibrium

Consider the case

Now, if \( A \) is a matrix with complex eigenvalues \( p \pm iq \), then with

eigenvectors \( \mathbf{v} \pm i\mathbf{w} \)
We showed that can show that
\[
\frac{d}{dt} x(t) = e^{Pt} \left[ \begin{array}{c} \cos \theta - \sin \theta \\ \sin \theta \cos \theta \end{array} \right] \left[ \begin{array}{c} \cos \phi \\ \sin \phi \end{array} \right] x(0)
\]

Then using argument similar to before

Then as \( t \to \infty \)

(1) \( p = 0 \), the trajectory is elliptical

(2) \( p > 0 \), the trajectory spirals outwards elliptically

(3) \( p < 0 \), spirals inwards exponentially

Note: If \( \frac{dx}{dt} = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix} x(t) \)

Then, for this example

Check \( \lambda = 1 \pm 2i \)

\[
\frac{d}{dt} x(t) = e^{t} \left[ \begin{array}{c} \cos 2t - \sin 2t \\ 2 \sin 2t \cos 2t \end{array} \right] x(0)
\]